Today 16.1

$\boxed{L16}$

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L16

Kinetics of a rigid body

L16

Today 16.1

Tuesday 16.1

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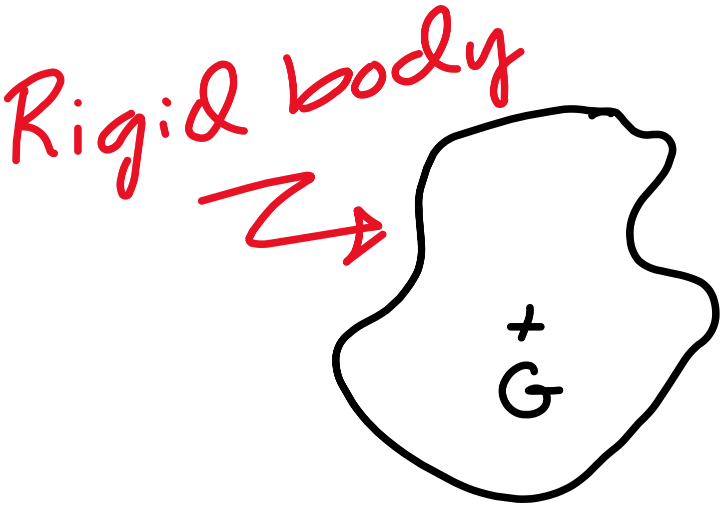
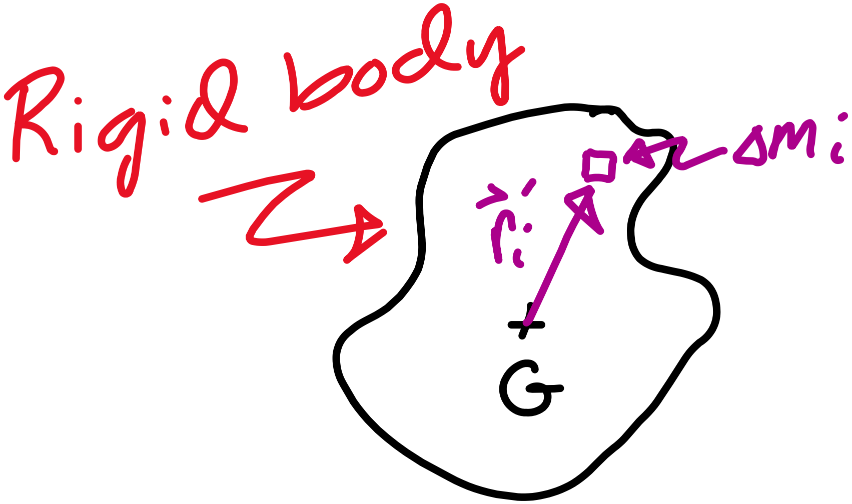
<u>Note</u>: We are going to take the rigid body as being made up of an ∞ number of point particles


Rigid body
G

We can write

$$\vec{H}_G = \sum_i (\vec{r}_i{}' \times \Delta m_i \vec{V}_i{}')$$



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For plane motion

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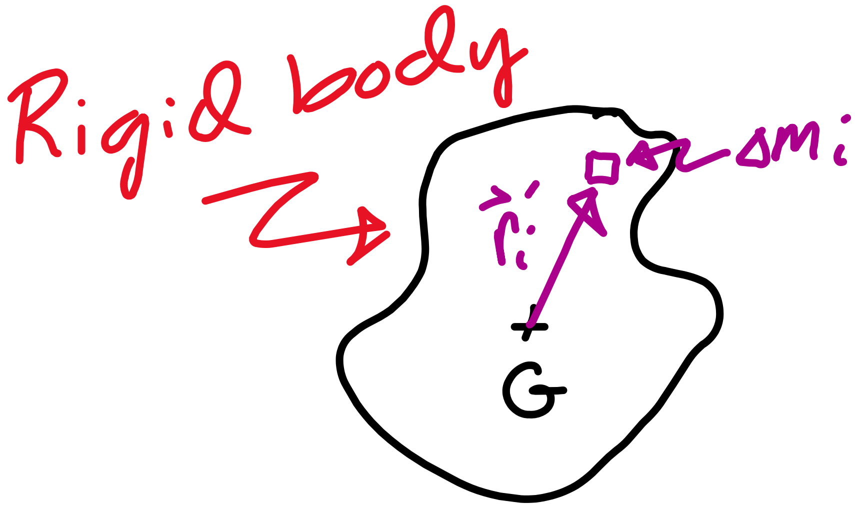

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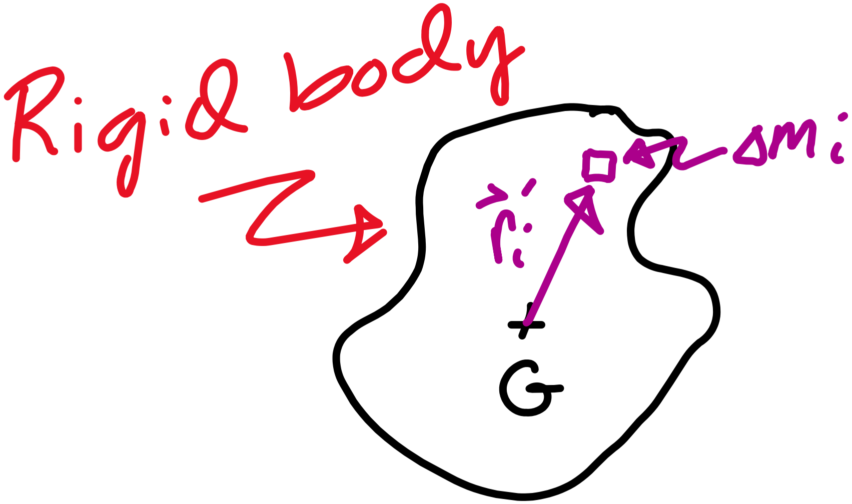

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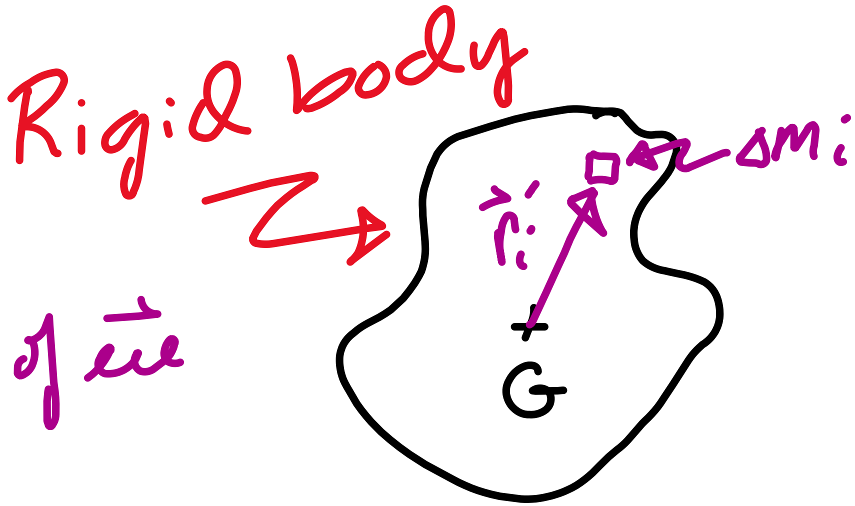

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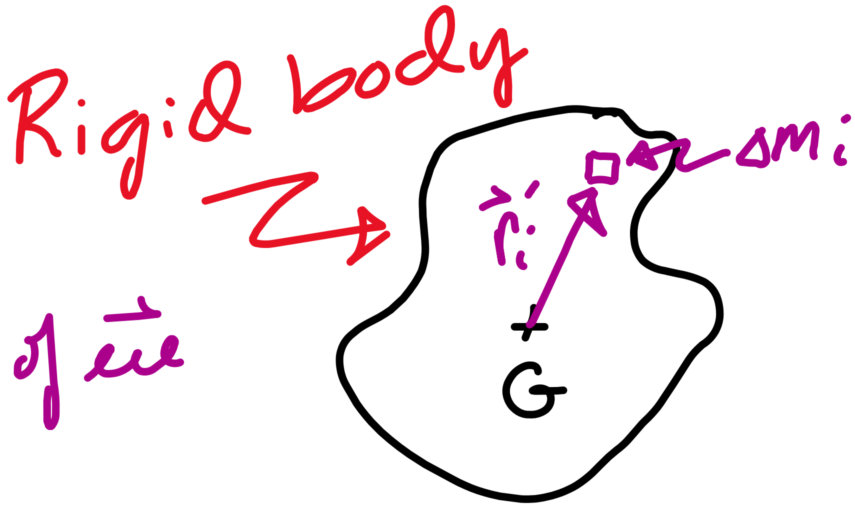

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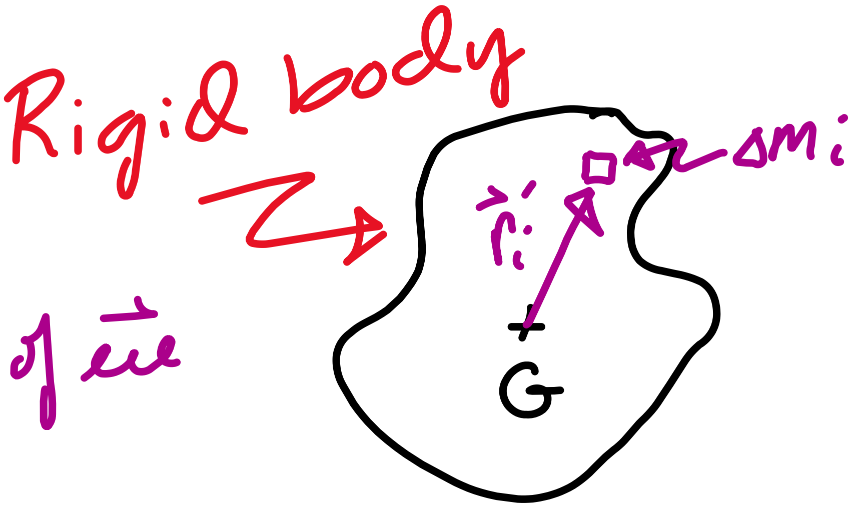

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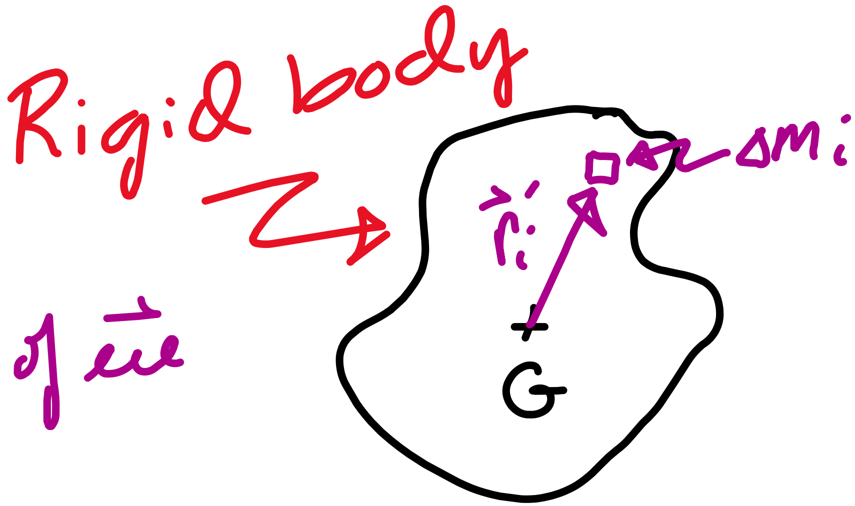

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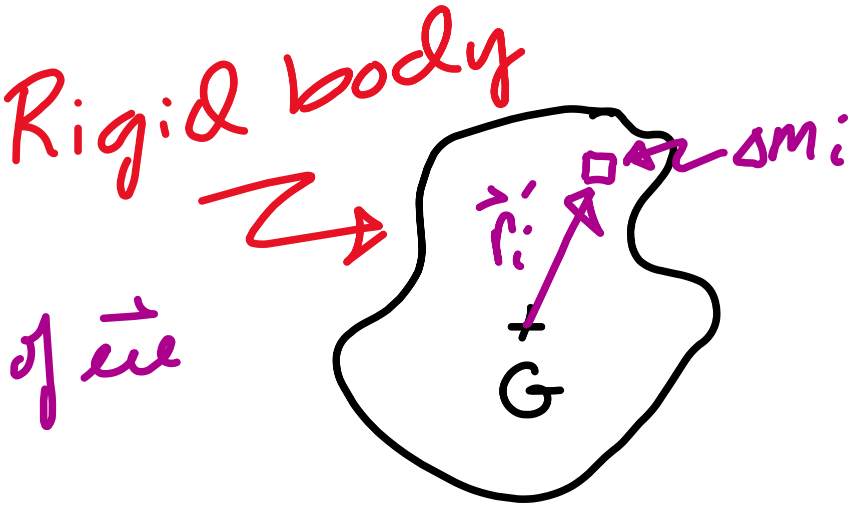

Define:

$\bar{I} \equiv$ Moment of inertia about G

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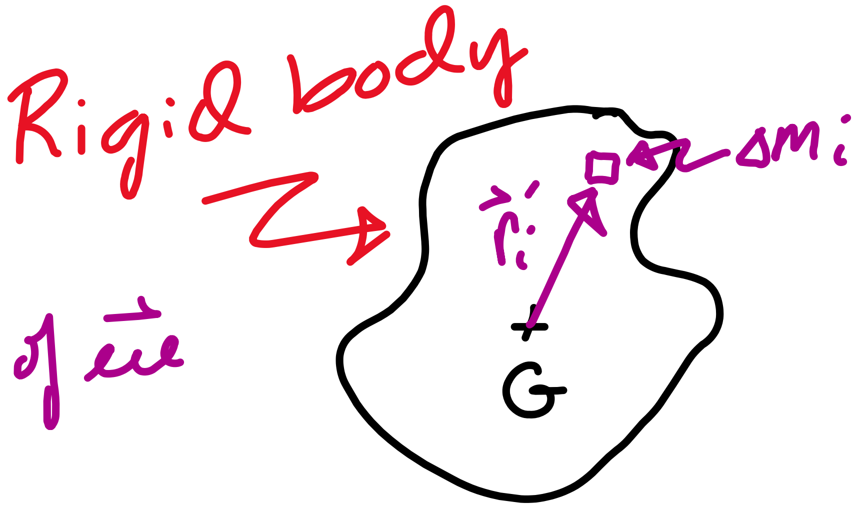

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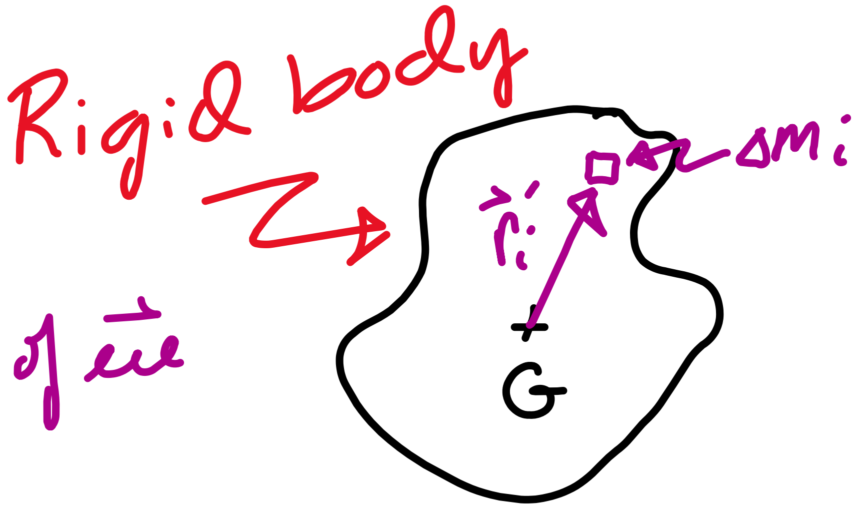

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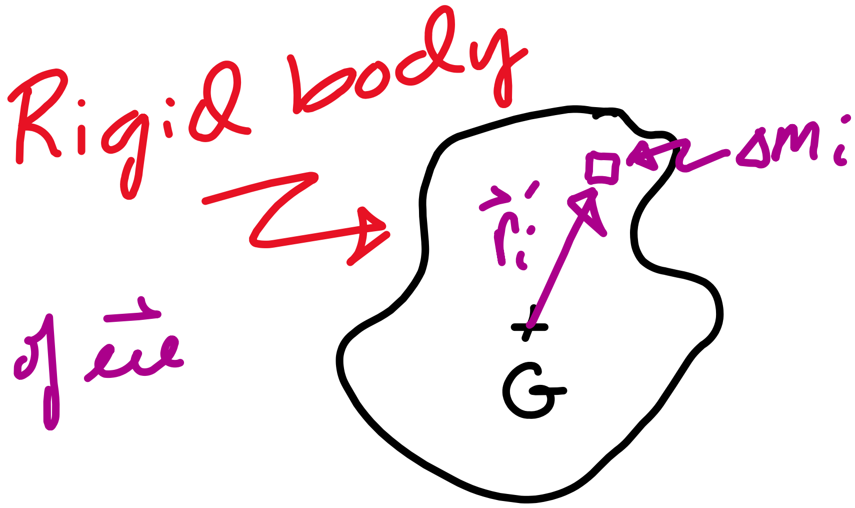

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If mass is uniform over 2-d surface

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If mass is uniform over 2-d surface with area A

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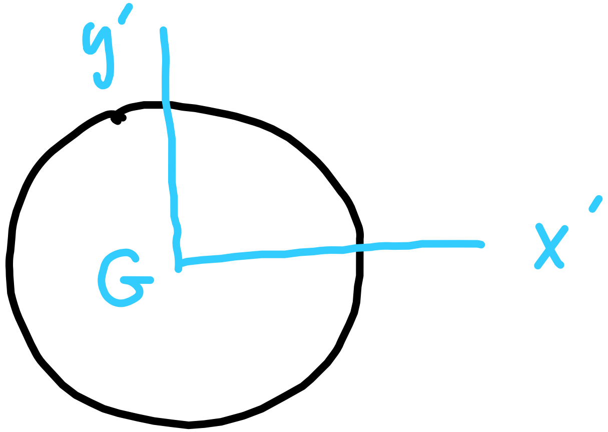
For polar coordinates

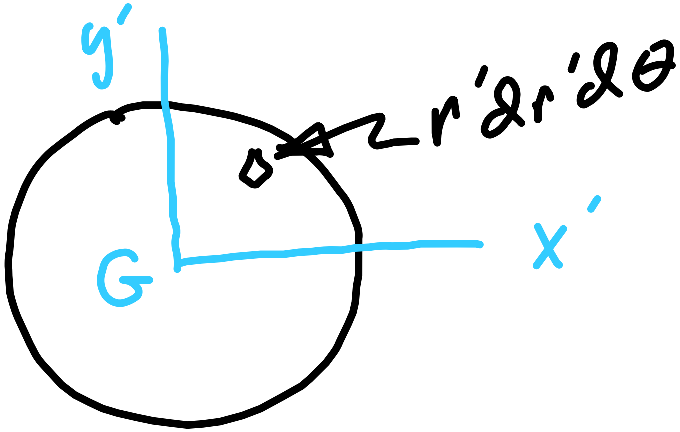
$$dx' dy' \to r' dr' d\theta'$$

Uniform disk

Uniform disk



Uniform disk



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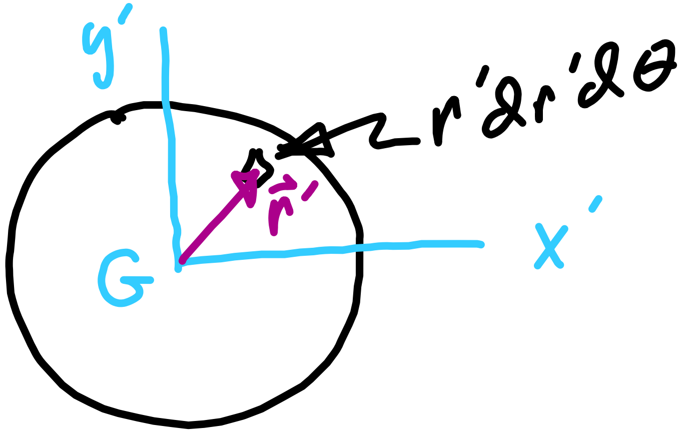
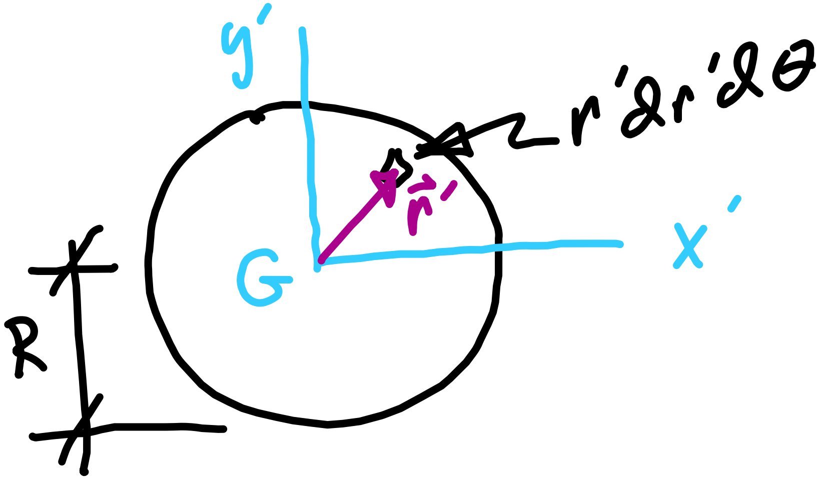
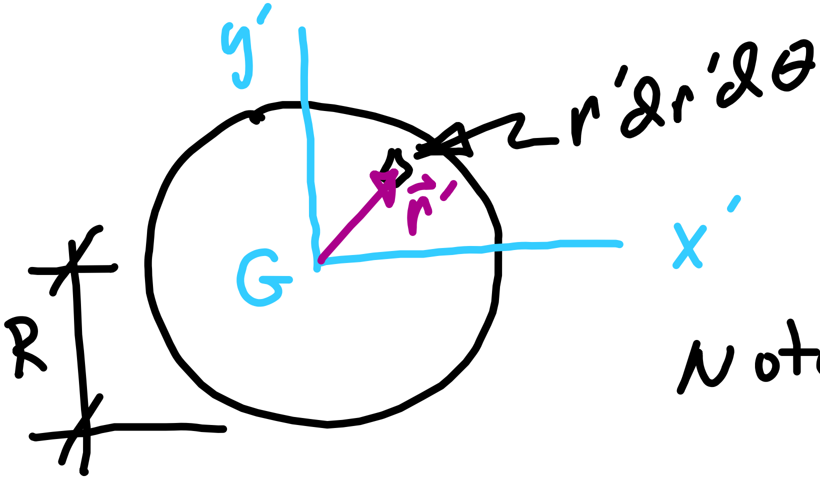


Uniform disk



$$\bar{I} = \left(\frac{M}{A}\right)\int_0^R\int_0^{2\pi}(r')^2\,r'dr'd\theta'$$

Uniform disk



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Note: $A = \pi R^2$

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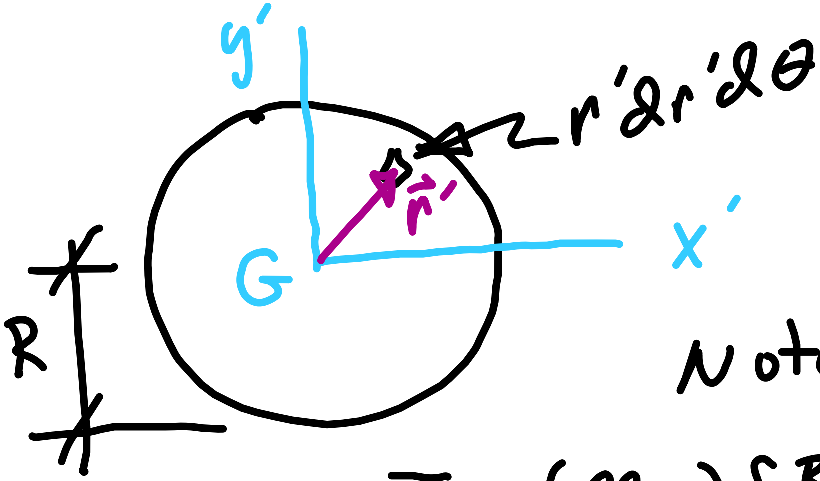

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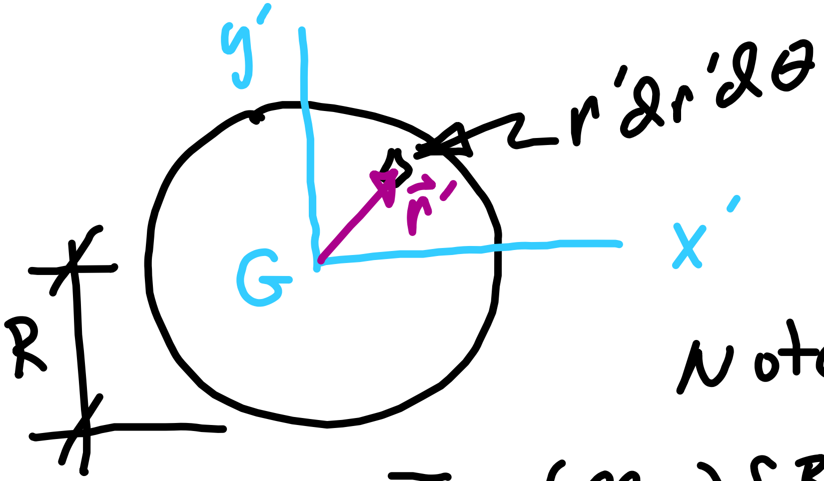

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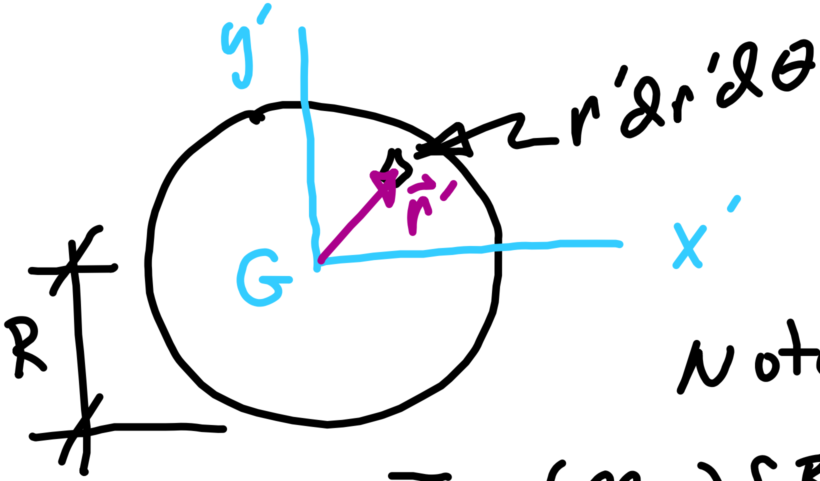
Uniform disk



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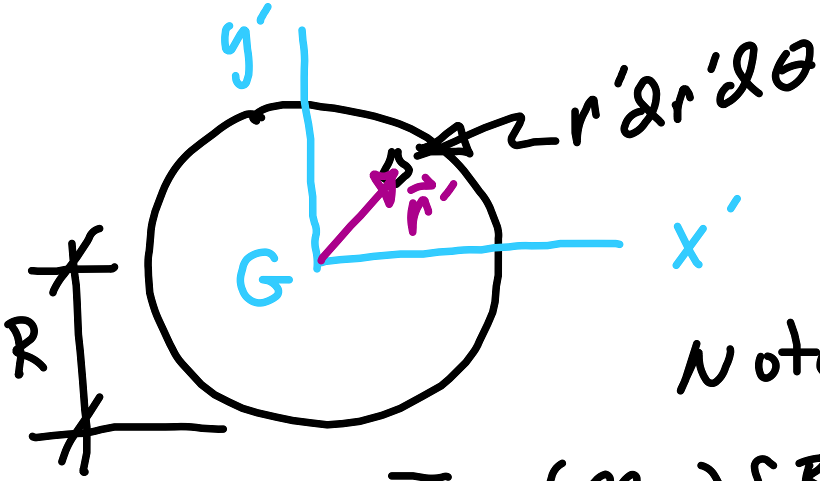

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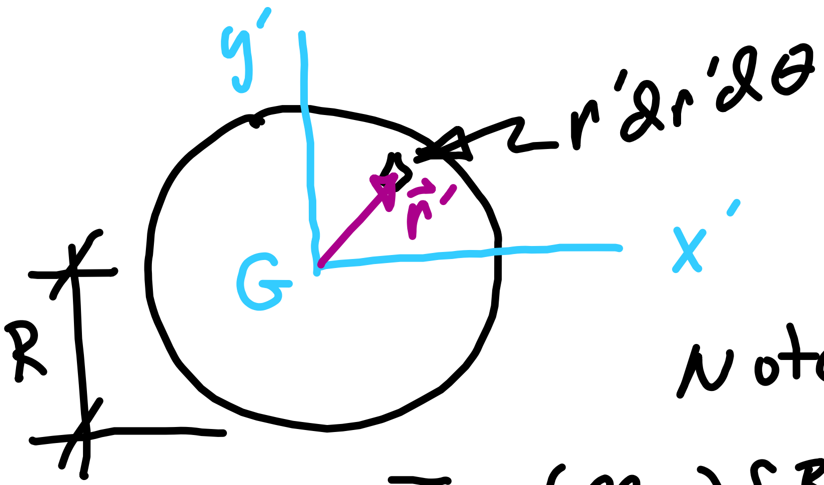

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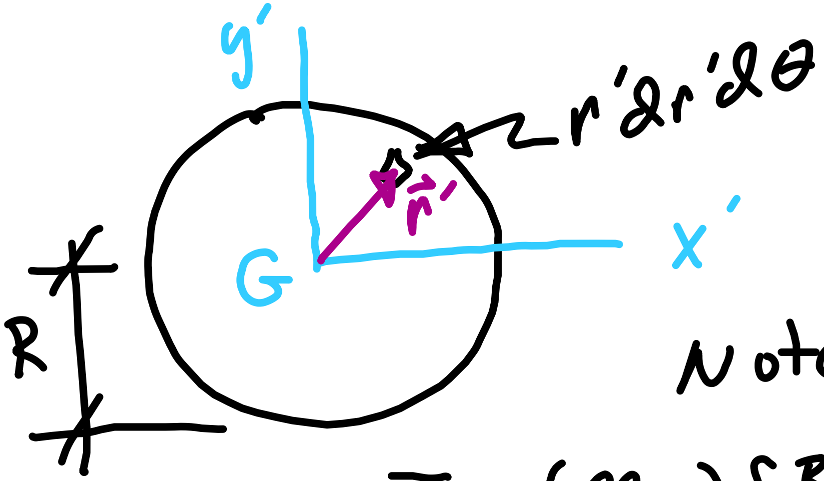

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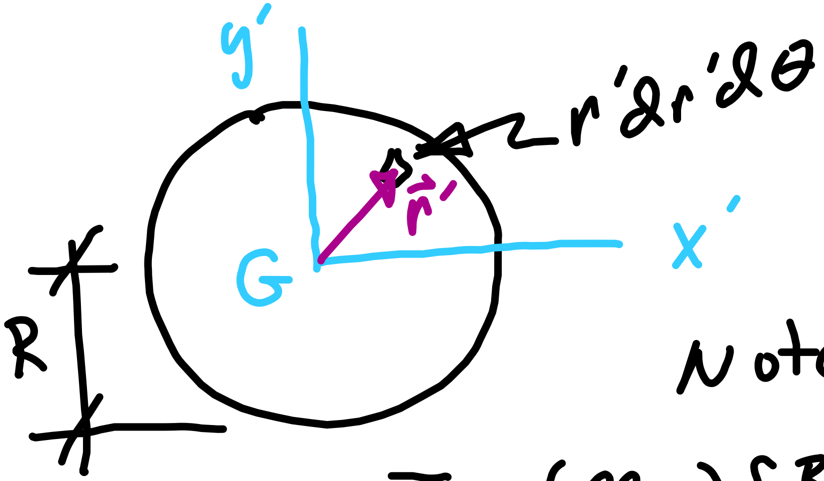

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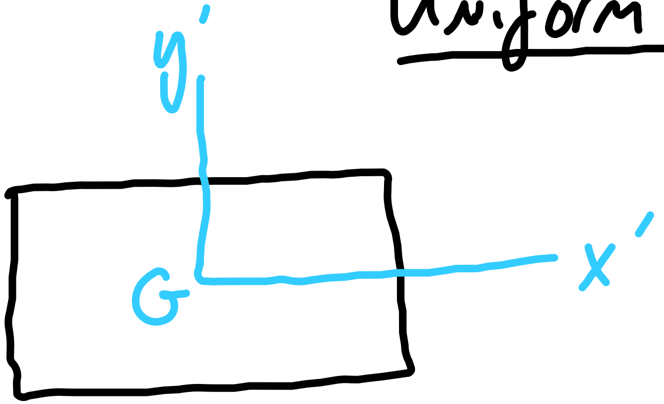
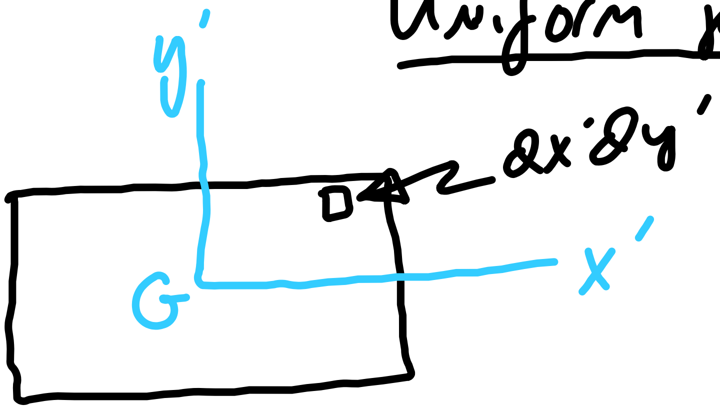
Uniform plate

Uniform plate



Uniform plate



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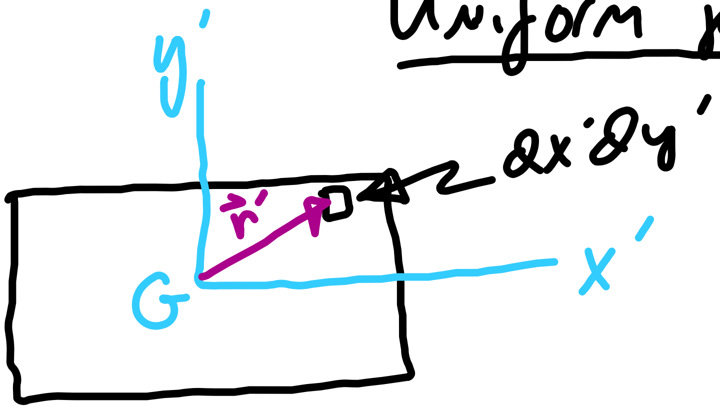

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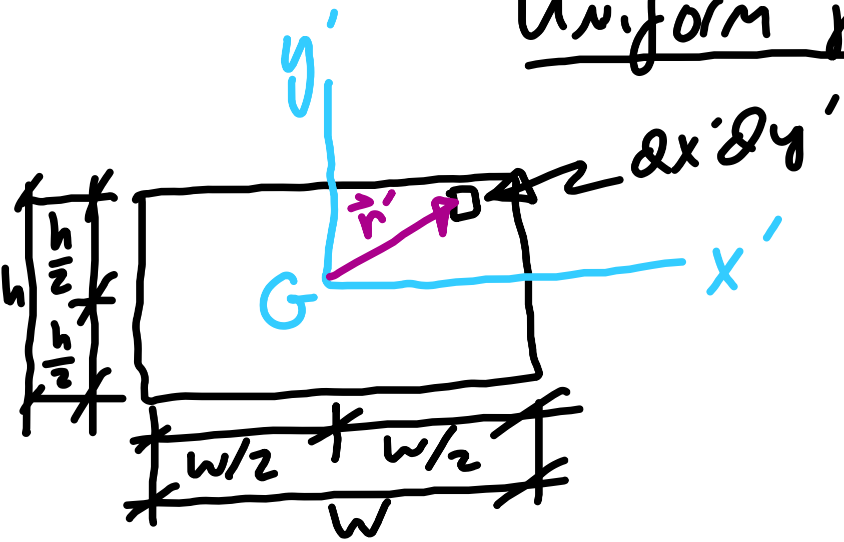

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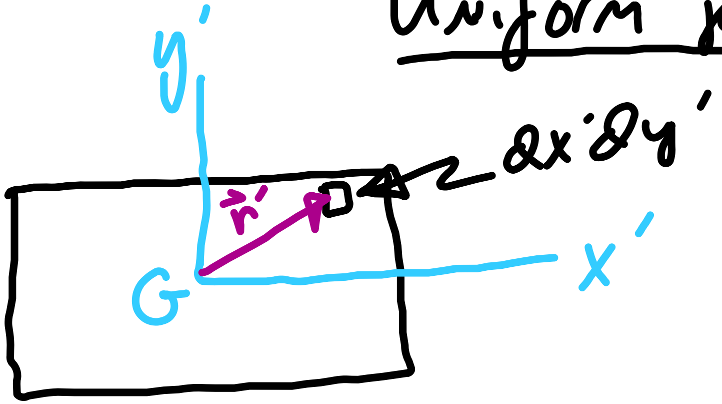
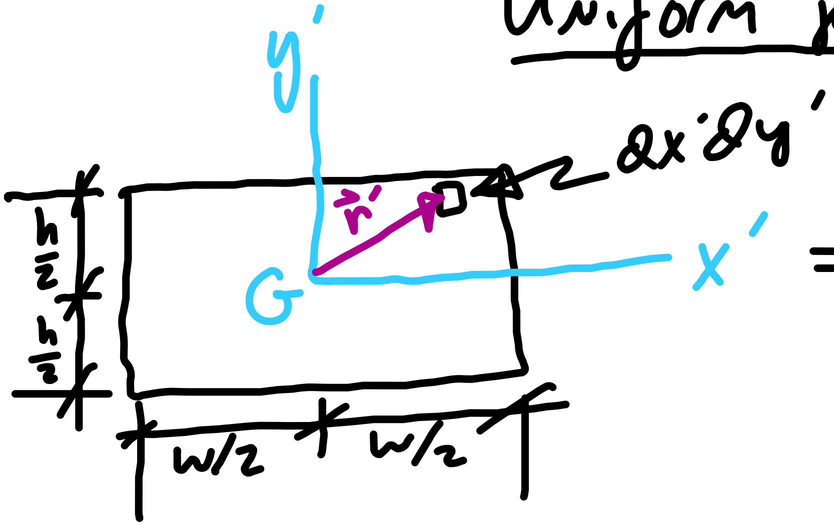


$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2 \, dx' \, dy'$$

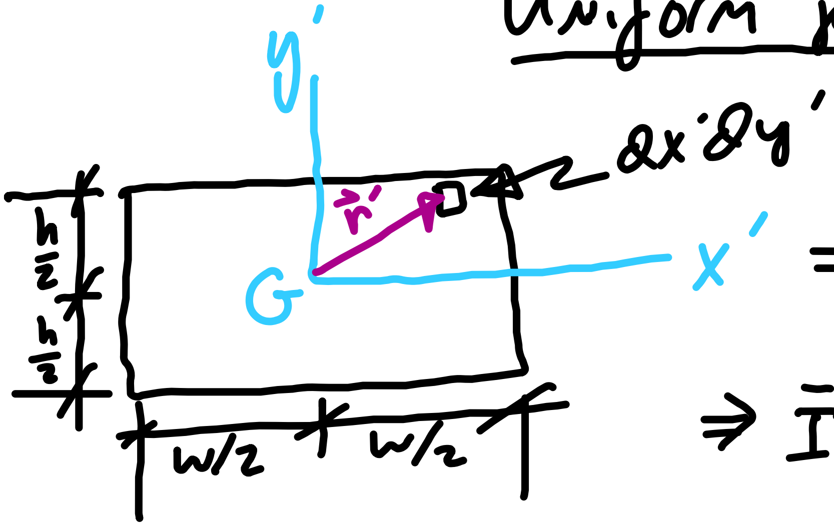
Uniform plate



$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2\,dx'\,dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'\,dy'$$

Uniform plate

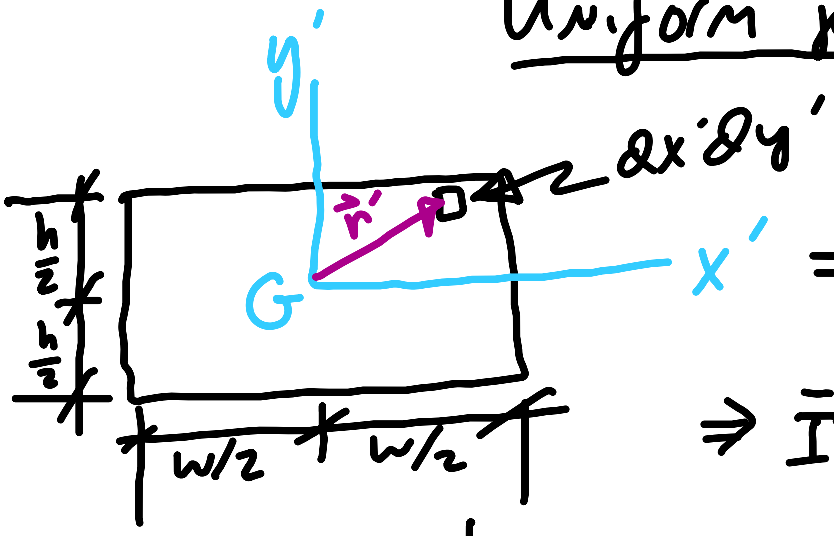


$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2\,dx'\,dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'\,dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{h/2}\left(\frac{x'^3}{3}+y'^2x'\right)\Big|_{-\frac{w}{2}}^{w/2}dy'$$

Uniform plate



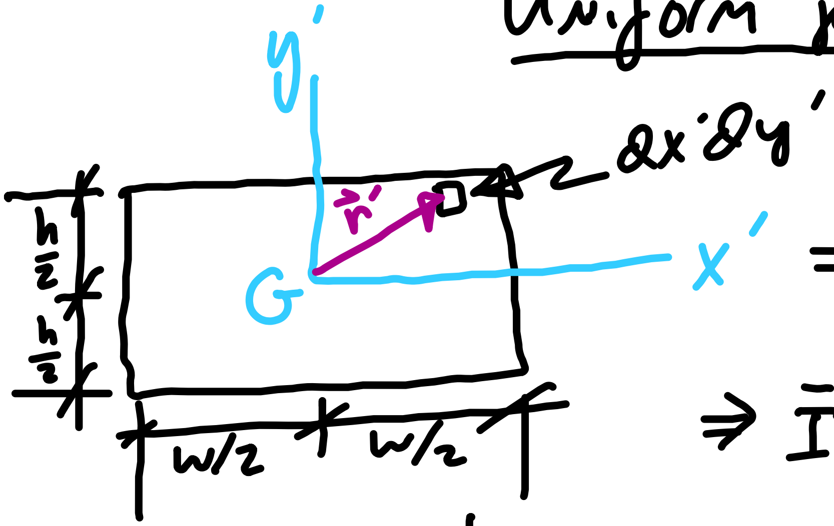
$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2\,dx'\,dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'\,dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{h/2}\left(\frac{x'^3}{3}+y'^2x'\right)\Bigg|_{-\frac{w}{2}}^{w/2}dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\frac{w^3}{24}+\frac{w}{2}y'^2-\left(-\frac{w^3}{24}\right)-\left(-\frac{w}{2}\right)y'^2\right]dy'$$

Uniform plate



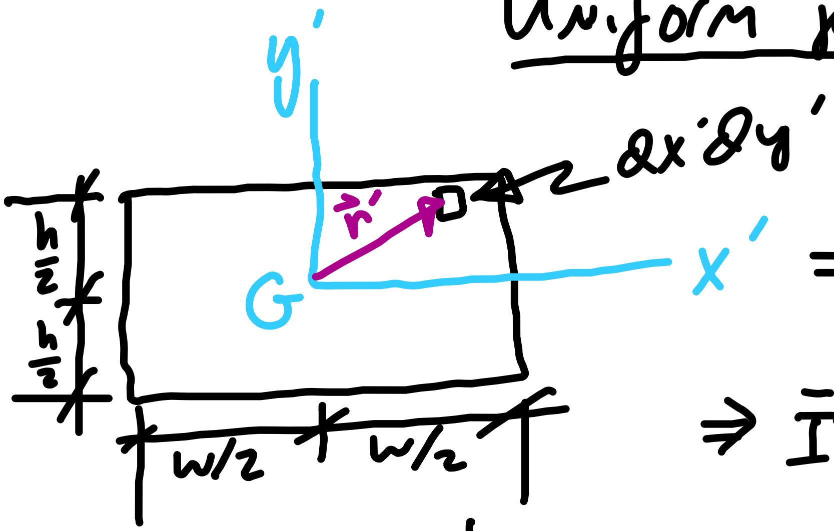
$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2 dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{h/2}\left(\frac{x'^3}{3}+y'^2x'\right)\Big|_{-\frac{w}{2}}^{w/2}dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(w^3/12 + wy'^2\right)dy'$$

Uniform plate



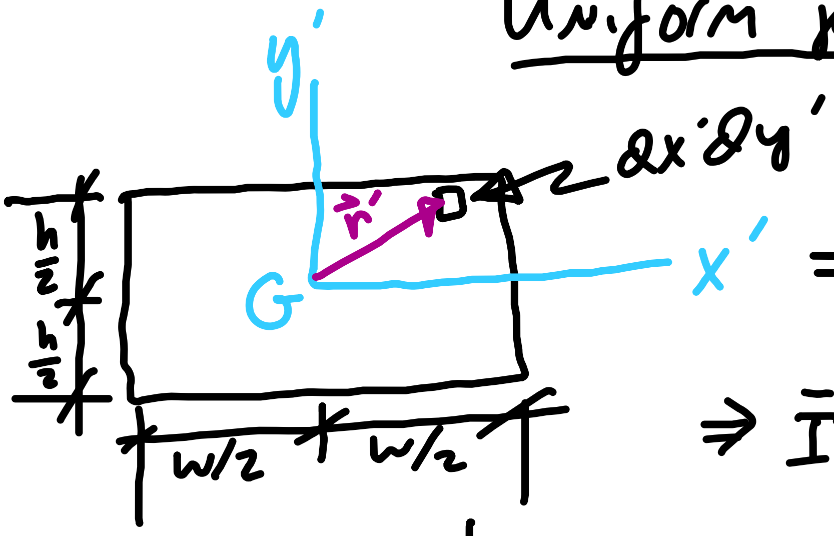
$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2\, dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{h/2}\left(\frac{x'^3}{3}+y'^2x'\right)\Big|_{-\frac{w}{2}}^{w/2} dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}(w^3/12 + wy'^2)\,dy' = \left(\frac{m}{A}\right)\left[\frac{w^3}{12}y' + \frac{wy'^3}{3}\right]\Big|_{-h/2}^{h/2}$$

Uniform plate



$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2\,dx'dy'$$

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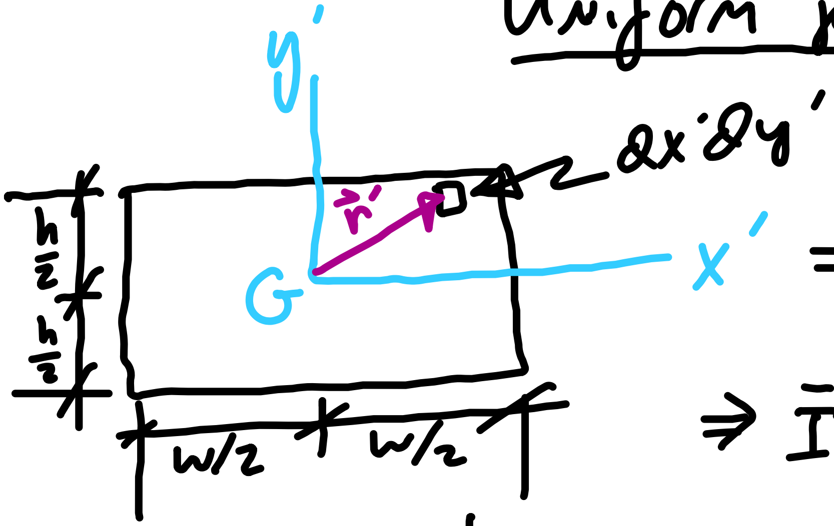
$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(w^3/12 + wy'^2\right)dy' = \left(\frac{m}{A}\right)\left[\frac{w^3}{12}y' + \frac{wy'^3}{3}\right]\Big|_{-h/2}^{h/2}$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\left[\frac{w^3}{12}\left(\frac{h}{2}\right)+\frac{w}{3}\left(\frac{h^3}{8}\right)-\frac{w^3}{12}\left(-\frac{h}{2}\right)-\frac{w}{3}\left(-\frac{h^3}{8}\right)\right]$$

Uniform plate



$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2\,dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}\left(x'^2+y'^2\right)dx'dy'$$

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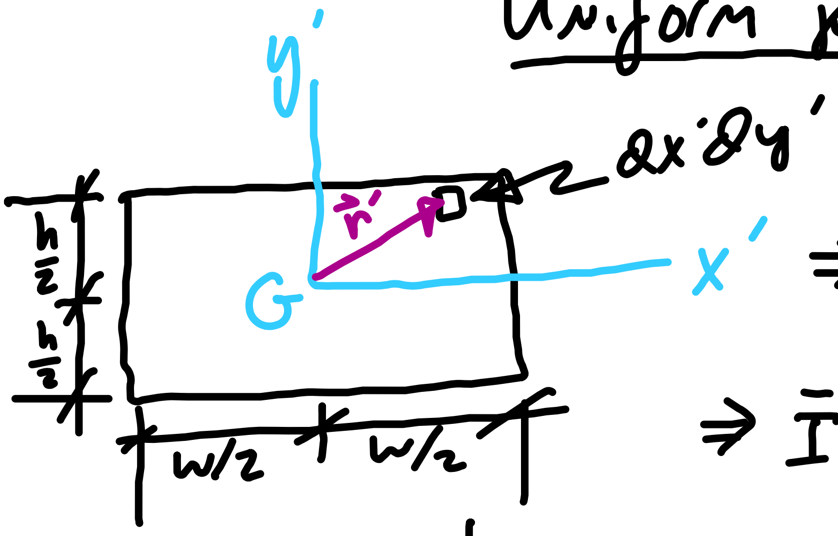
$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\left[\frac{w^3h}{12} + \frac{wh^3}{12}\right]$$

Uniform plate



$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2 \, dx'dy'$$

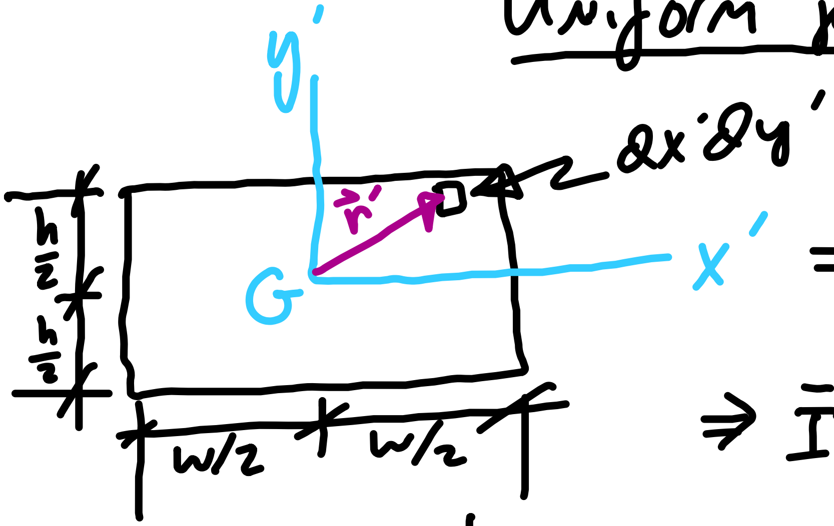
$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{h/2}\left(\frac{x'^3}{3}+y'^2x'\right)\Big|_{-\frac{w}{2}}^{w/2} dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(w^3/12 + wy'^2\right)dy' = \left(\frac{m}{A}\right)\left[\frac{w^3}{12}y' + \frac{wy'^3}{3}\right]\Big|_{-h/2}^{h/2}$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\left[\frac{w^3h}{12}+\frac{wh^3}{12}\right] = \left(\frac{m}{A}\right)\left(\frac{wh}{12}\right)\left[w^2+h^2\right]$$

Uniform plate



$$\bar{I} = \left(\frac{M}{A}\right)\iint r'^2\,dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{M}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'dy'$$

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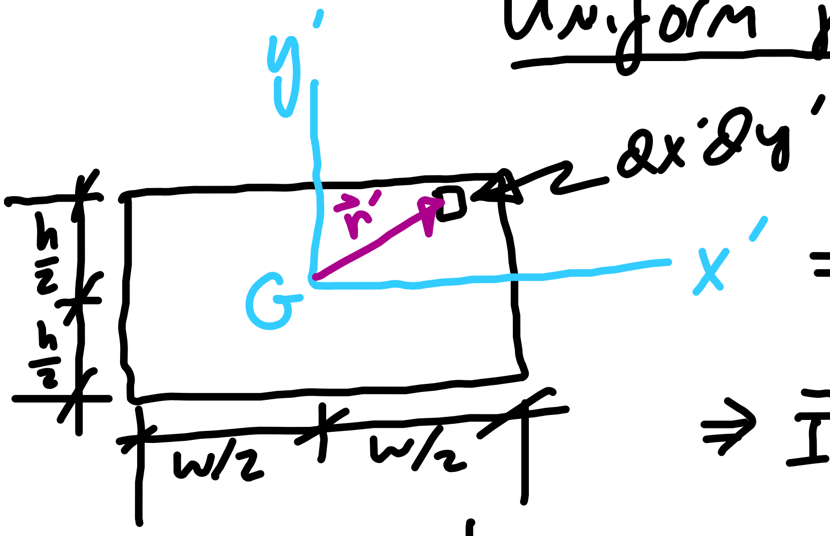
$$\Rightarrow \bar{I} = \left(\frac{M}{A}\right)\left[\frac{w^3h}{12} + \frac{wh^3}{12}\right] = \left(\frac{M}{A}\right)\left(\frac{wh}{12}\right)\left[w^2+h^2\right]$$

But $A = wh$

Uniform plate



$$\bar{I} = \left(\frac{M}{A}\right)\iint r'^2\, dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{M}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'dy'$$

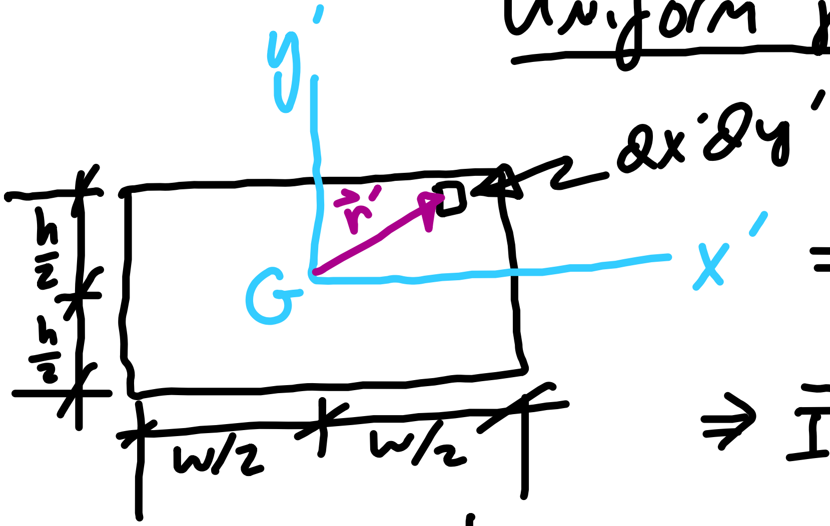
$$\Rightarrow \bar{I} = \left(\frac{M}{A}\right)\int_{-\frac{h}{2}}^{h/2}\left(\frac{x'^3}{3}+y'^2x'\right)\Big|_{-\frac{w}{2}}^{w/2} dy'$$

$$\Rightarrow \bar{I} = \left(\frac{M}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(w^3/12 + wy'^2\right)dy' = \left(\frac{M}{A}\right)\left[\frac{w^3}{12}y' + \frac{wy'^3}{3}\right]\Big|_{-h/2}^{h/2}$$

$$\Rightarrow \bar{I} = \left(\frac{M}{A}\right)\left[\frac{w^3h}{12} + \frac{wh^3}{12}\right] = \left(\frac{M}{A}\right)\left(\frac{wh}{12}\right)\left[w^2+h^2\right]$$

But $A = wh$ so $\boxed{\bar{I} = \frac{M}{12}\left[w^2+h^2\right]}$

Uniform plate



$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2\,dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{h/2}\left(\frac{x'^3}{3}+y'^2x'\right)\Big|_{-\frac{w}{2}}^{w/2}dy'$$

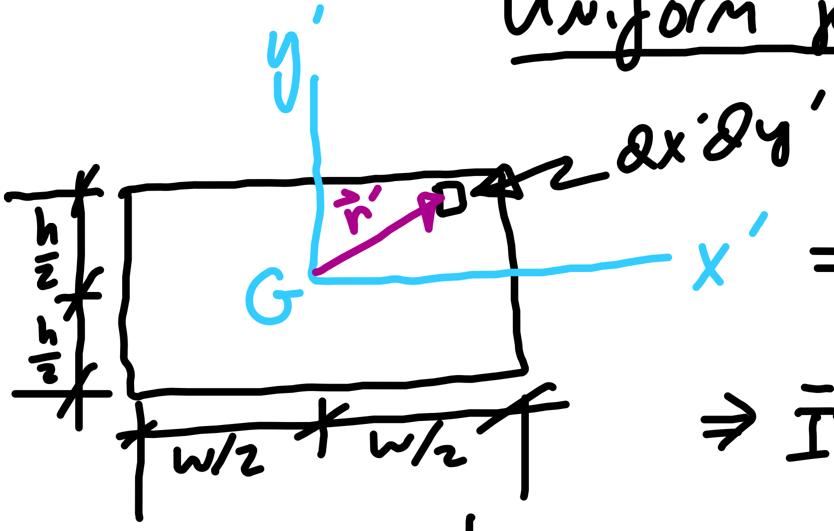
$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}(w^3/12+wy'^2)\,dy' = \left(\frac{m}{A}\right)\left[\frac{w^3}{12}y'+\frac{wy'^3}{3}\right]\Big|_{-h/2}^{h/2}$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\left[\frac{w^3h}{12}+\frac{wh^3}{12}\right] = \left(\frac{m}{A}\right)\left(\frac{wh}{12}\right)[w^2+h^2]$$

But $A = wh$ so $\boxed{\bar{I} = \frac{m}{12}[w^2+h^2]}$

& $\vec{H}_G = \bar{I}\vec{\omega}$

Uniform plate



$$\bar{I} = \left(\frac{m}{A}\right)\iint r'^2\,dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}(x'^2+y'^2)\,dx'dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{h/2}\left(\frac{x'^3}{3}+y'^2x'\right)\Big|_{-\frac{w}{2}}^{w/2}dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}}(w^3/12 + wy'^2)\,dy' = \left(\frac{m}{A}\right)\left[\frac{w^3}{12}y' + \frac{wy'^3}{3}\right]\Big|_{-h/2}^{h/2}$$

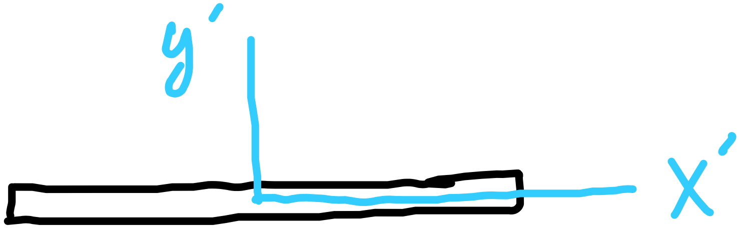
$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\left[\frac{w^3h}{12} + \frac{wh^3}{12}\right] = \left(\frac{m}{A}\right)\left(\frac{wh}{12}\right)[w^2+h^2]$$

But $A = wh$ so $\boxed{\bar{I} = \frac{m}{12}[w^2+h^2]}$

& $\vec{H}_G = \bar{I}\vec{\omega} \Rightarrow \boxed{\vec{H}_G = \left(\frac{m}{12}\right)[w^2+h^2]\vec{\omega}}$

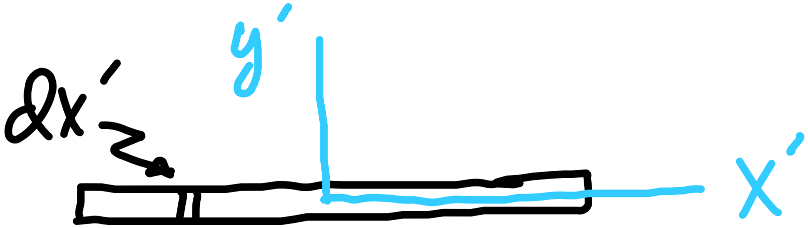
Slender rod

Slender rod



Only need 1D
in this case

Slender rod



Only need 1d in this case

Slender rod

Only need 1d in this case

Slender rod

Only need 1d in this case

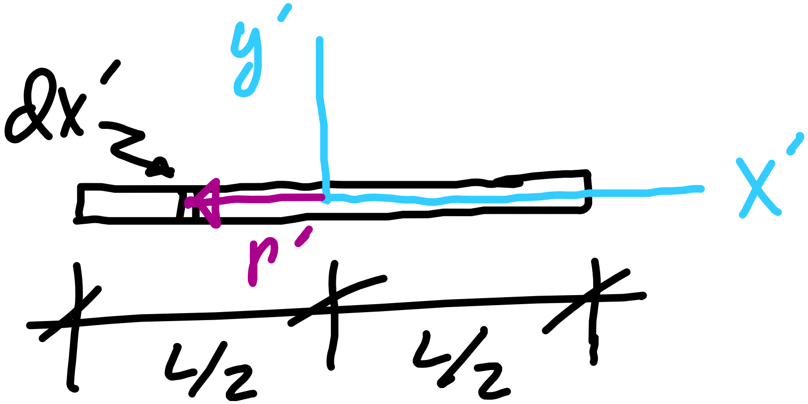
Here $\frac{M}{A}$

Slender rod

Only need 1d in this case

Here $\frac{M}{A} \rightarrow \frac{M}{L}$

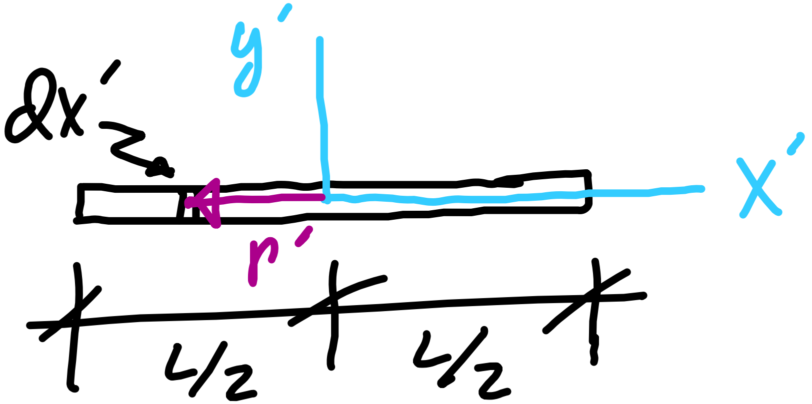
Slender rod



Only need 1D in this case

Here $\frac{M}{A} \rightarrow \frac{m}{L}$

Slender rod



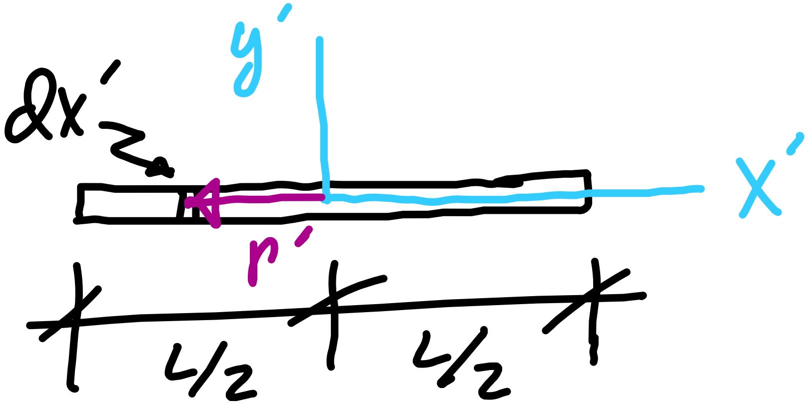
Only need 1d in this case

Here $\frac{M}{A} \rightarrow \frac{m}{L}$

Now $\bar{I} = \left(\frac{m}{L}\right)\int_{-L/2}^{L/2} x'^2 \, dx'$

Slender rod

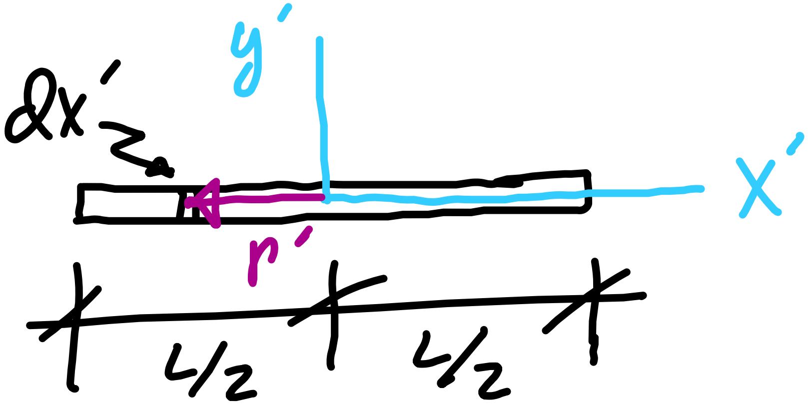
Only need 1D in this case



Here $\frac{M}{A} \rightarrow \frac{M}{L}$

Now $\bar{I} = \left(\frac{M}{L}\right)\int_{-L/2}^{L/2} x'^2 \, dx' = \left(\frac{M}{L}\right)\left[\frac{x'^3}{3}\right]\Big|_{-L/2}^{L/2}$

Slender rod



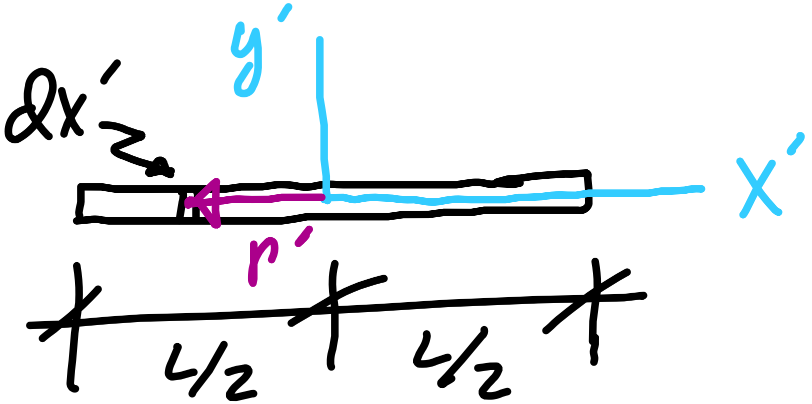
Only need 1d in this case

Here $\frac{M}{A} \rightarrow \frac{M}{L}$

Now $I = \left(\frac{M}{L}\right)\int_{-L/2}^{L/2} x'^2 \, dx' = \left(\frac{M}{L}\right)\left[\frac{x'^3}{3}\right]\Bigg|_{-L/2}^{L/2}$

$\Rightarrow \quad I = \left(\frac{M}{3L}\right)\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right]$

Slender rod



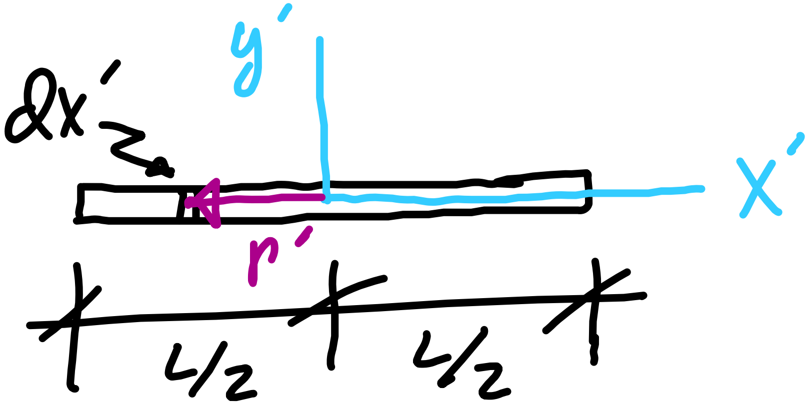
Only need 1d in this case

Here $\frac{M}{A} \rightarrow \frac{M}{L}$

Now $\bar{I} = \left(\frac{M}{L}\right)\int_{-L/2}^{L/2} x'^2 \, dx' = \left(\frac{M}{L}\right)\left[\frac{x'^3}{3}\right]\Big|_{-L/2}^{L/2}$

$\Rightarrow \bar{I} = \left(\frac{M}{3L}\right)\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right] \Rightarrow \boxed{\bar{I} = \frac{ML^2}{12}}$

Slender rod



Only need 1D in this case

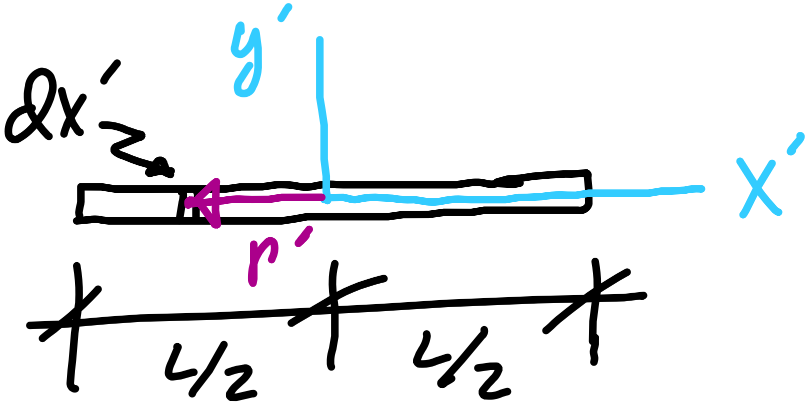
Here $\frac{M}{A} \rightarrow \frac{M}{L}$

Now $\bar{I} = \left(\frac{M}{L}\right)\int_{-L/2}^{L/2} x'^2 dx' = \left(\frac{M}{L}\right)\left[\frac{x'^3}{3}\right]\Big|_{-L/2}^{L/2}$

$\Rightarrow \quad \bar{I} = \left(\frac{M}{3L}\right)\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right] \Rightarrow \boxed{\bar{I} = \frac{ML^2}{12}}$

Compare to plate

Slender rod



Only need 1D in this case

Here $\frac{M}{A} \rightarrow \frac{M}{L}$

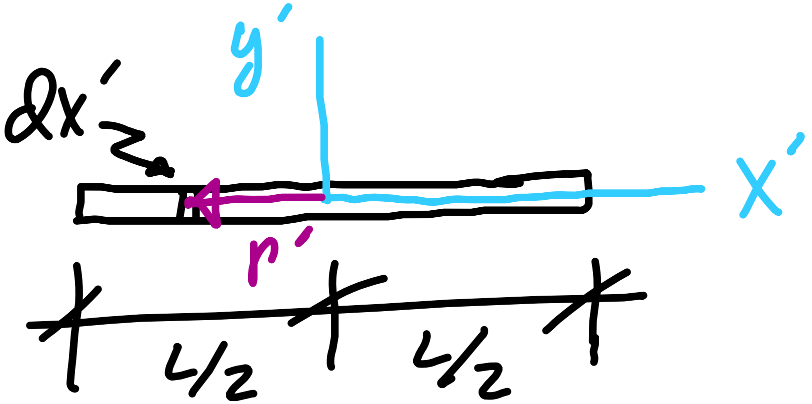
Now $\bar{I} = \left(\frac{M}{L}\right)\int_{-L/2}^{L/2} x'^2 \, dx' = \left(\frac{M}{L}\right)\left[\frac{x'^3}{3}\right]\Bigg|_{-L/2}^{L/2}$

$\Rightarrow \quad \bar{I} = \left(\frac{M}{3L}\right)\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right] \Rightarrow \boxed{\bar{I} = \frac{ML^2}{12}}$

Compare to plate

$\bar{I}_{plate} = \left(\frac{M}{12}\right)\left[L^2 + w^2\right]$

Slender rod



Only need 1D in this case

Here $\frac{M}{A} \rightarrow \frac{M}{L}$

Now $\bar{I} = \left(\frac{M}{L}\right)\int_{-L/2}^{L/2} x'^2 \, dx' = \left(\frac{M}{L}\right)\left[\frac{x'^3}{3}\right]\Bigg|_{-L/2}^{L/2}$

$\Rightarrow \quad \bar{I} = \left(\frac{M}{3L}\right)\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right] \Rightarrow \boxed{\bar{I} = \frac{ML^2}{12}}$

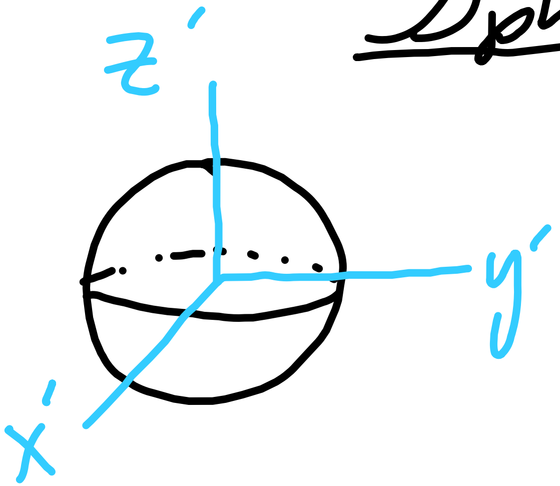
Compare to plate

$I_{plate} = \left(\frac{M}{12}\right)\left[L^2 + w^2\right]$

Slender rod is just a plate with $w \rightarrow 0$

Sphere

Sphere



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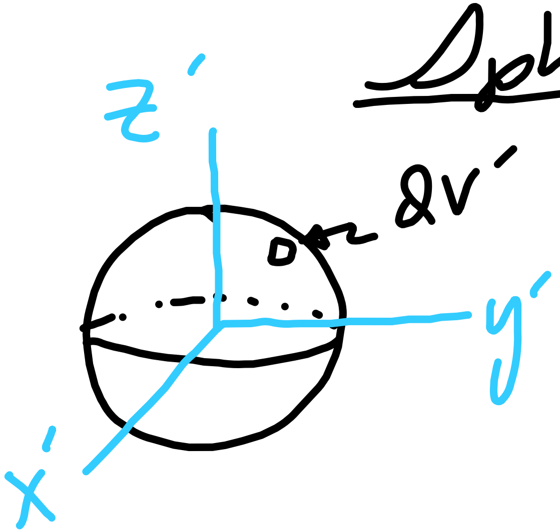
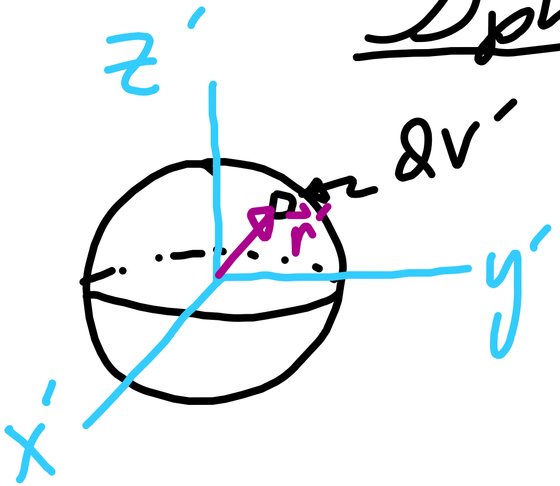


Here we need 3d

where $dv' = (dr')(r'\sin\phi' d\phi)(r' d\theta)$

Sphere

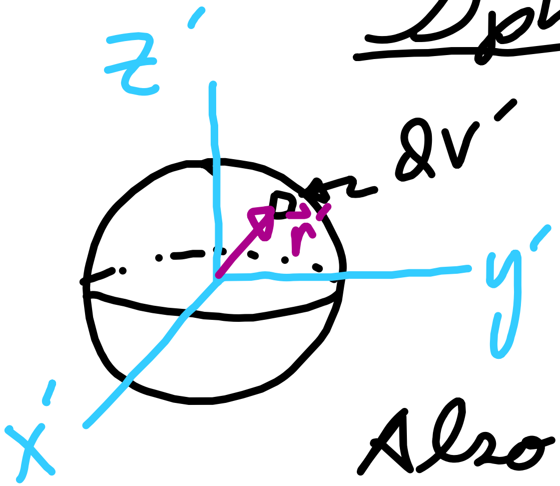


Here we need 3d

where $dv' = (dr')(r'\sin\varphi' d\varphi')(r' d\theta')$

$\Rightarrow dv' = r'^2 \sin\varphi' dr' d\varphi' d\theta'$

Sphere



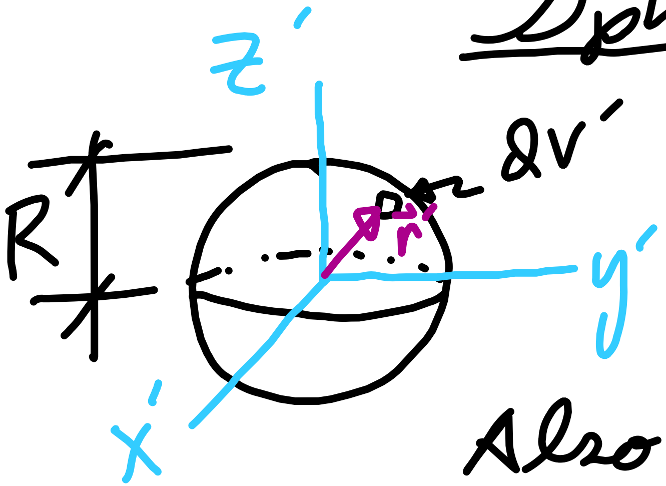
Here we need 3d

where $dv' = (dr')(r'\sin\phi' d\phi')(r' d\theta')$

$\Rightarrow dv' = r'^2 \sin\phi' dr' d\phi' d\theta'$

Also $\frac{M}{A} \to \frac{M}{V}$

Sphere



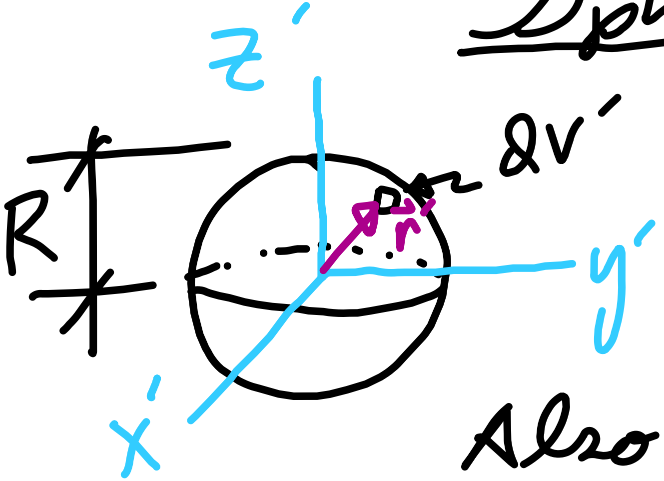
Here we need 3d

where $dV' = (dr')(r'\sin\phi' d\phi')(r' d\theta')$

$\Rightarrow dV' = r'^2 \sin\phi' \, dr' \, d\phi' \, d\theta'$

Also $\frac{M}{A} \to \frac{M}{V}$ & $V = \frac{4}{3}\pi R^3$

Sphere



Here we need 3d

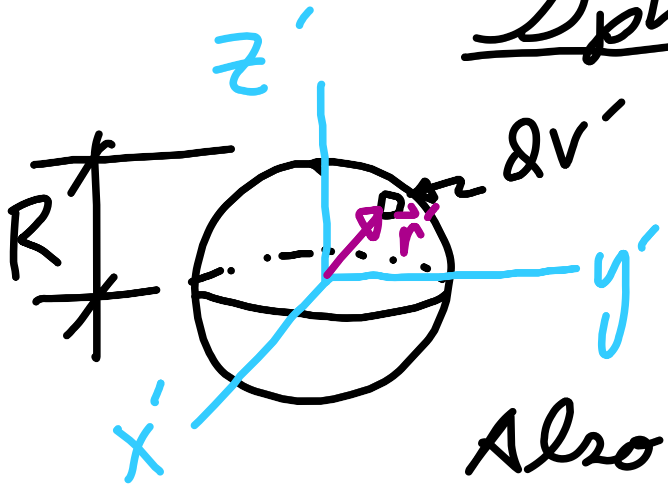
where $dV' = (dr')(r'\sin\phi' d\phi')(r' d\theta')$

$\Rightarrow dV' = r'^2 \sin\phi' dr' d\phi' d\theta'$

Also $\frac{M}{A} \to \frac{M}{V}$ & $V = \frac{4}{3}\pi R^3$

Another modification is that we don't want to integrate over r'^2

Sphere



Here we need 3d

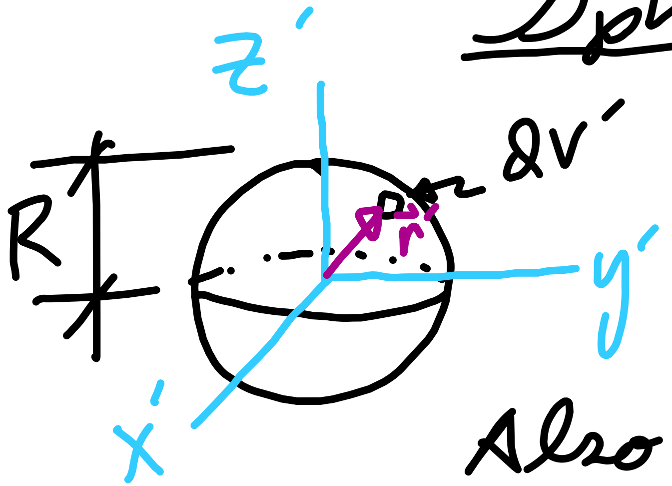
where $dV' = (dr')(r'\sin\phi' d\phi')(r' d\theta')$

$$\Rightarrow dV' = r'^2 \sin\phi' dr' d\phi' d\theta'$$

Also $\frac{M}{A} \to \frac{M}{V}$ & $V = \frac{4}{3}\pi R^3$

Another modification is that we don't want to integrate over r'^2 [as we do for 2d case]

Sphere



Here we need 3d

where $dV' = (dr')(r'\sin\phi' d\phi')(r' d\theta')$

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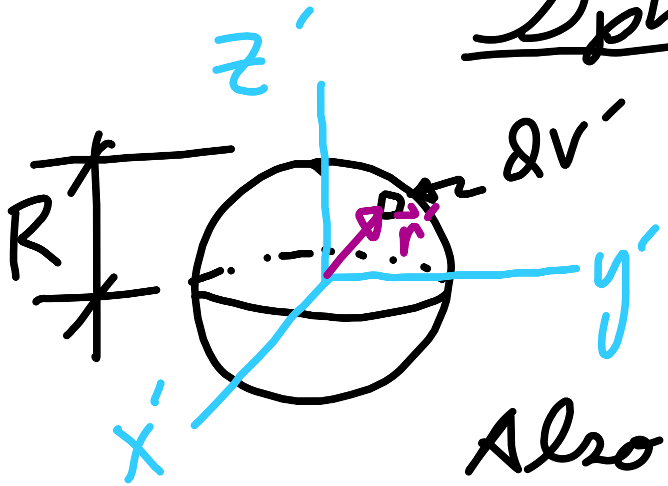

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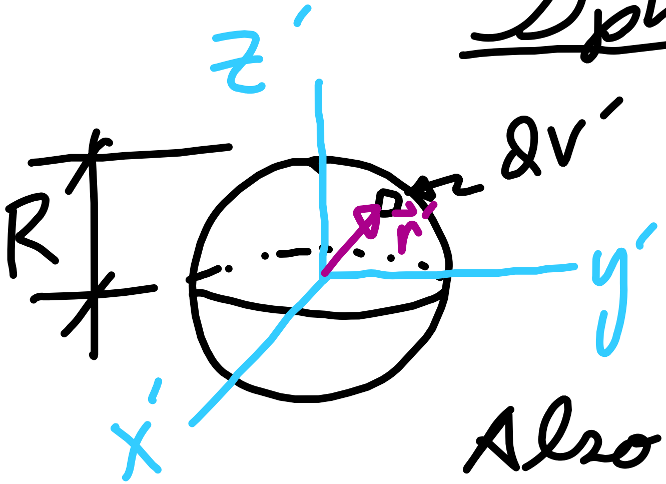
Sphere



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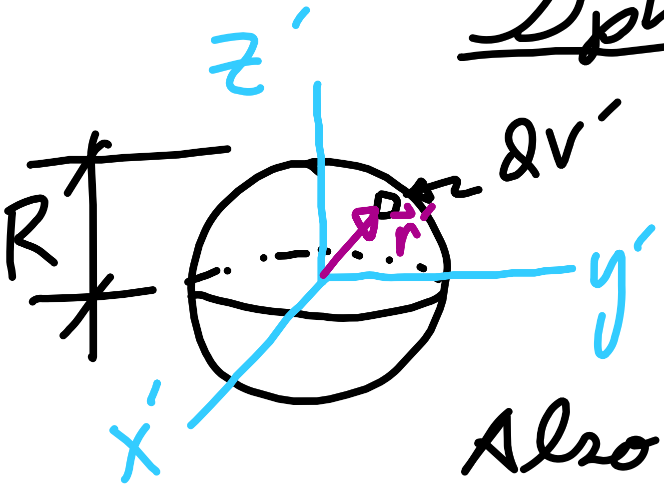

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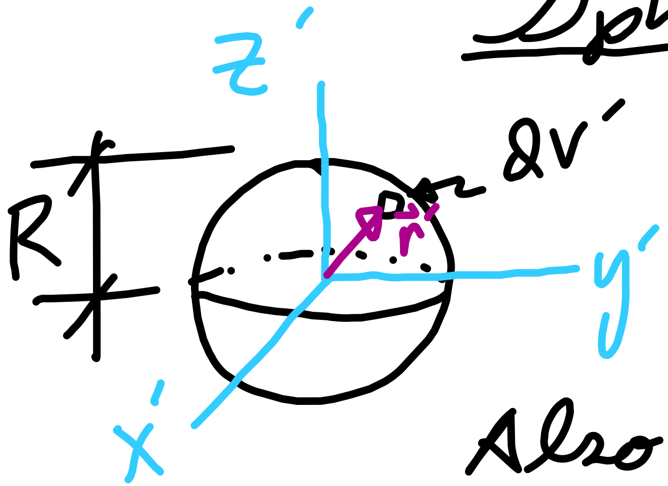

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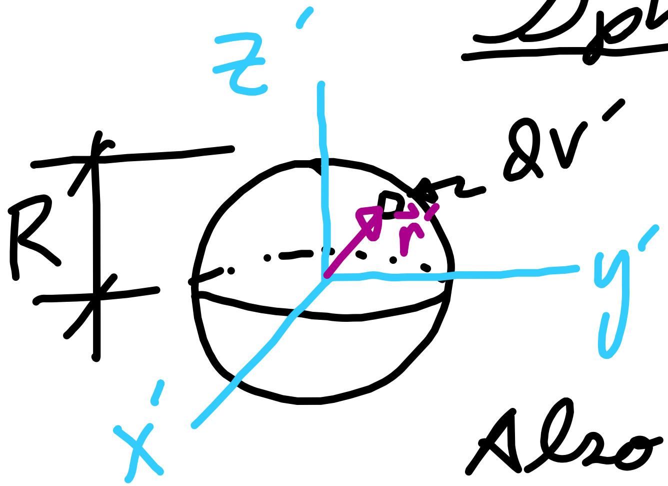
Used Wolfram alpha to find $\int_0^\pi \sin^3\varphi \, d\varphi = 4/3$

Sphere



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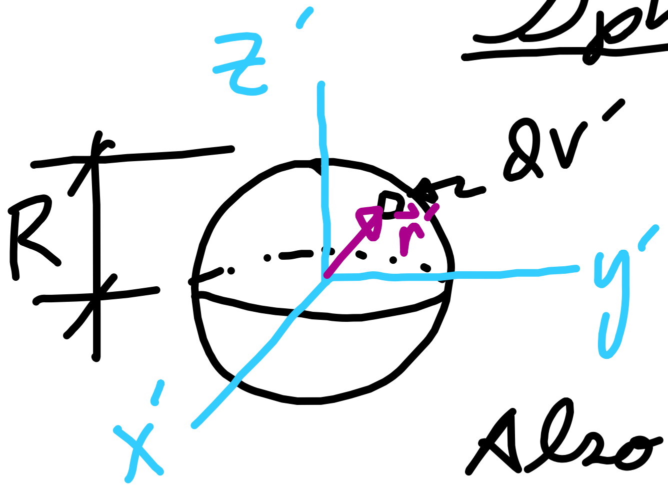
Sphere



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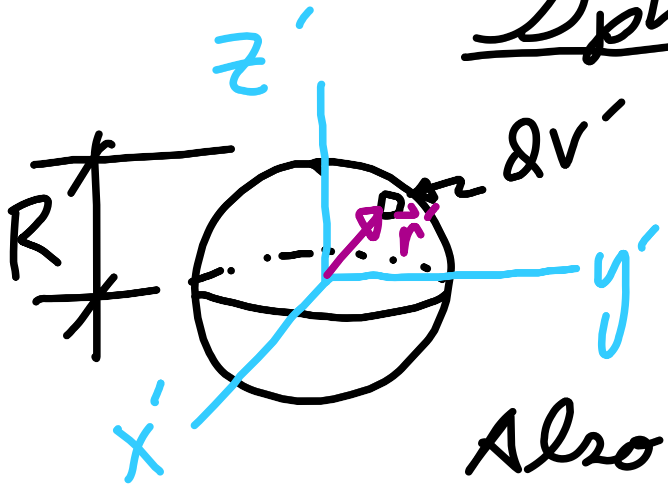
Sphere



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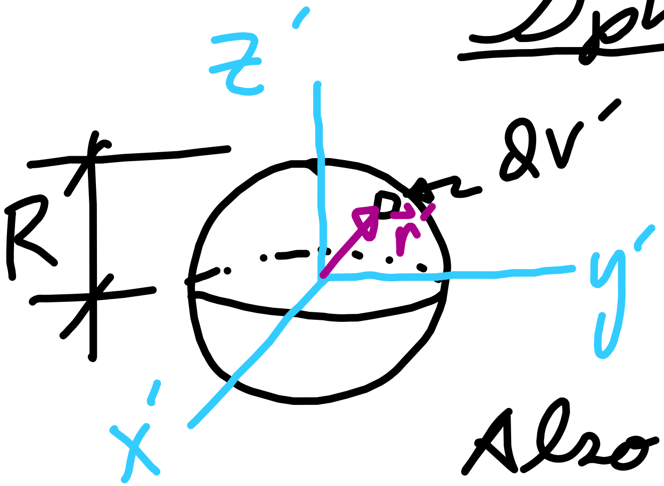

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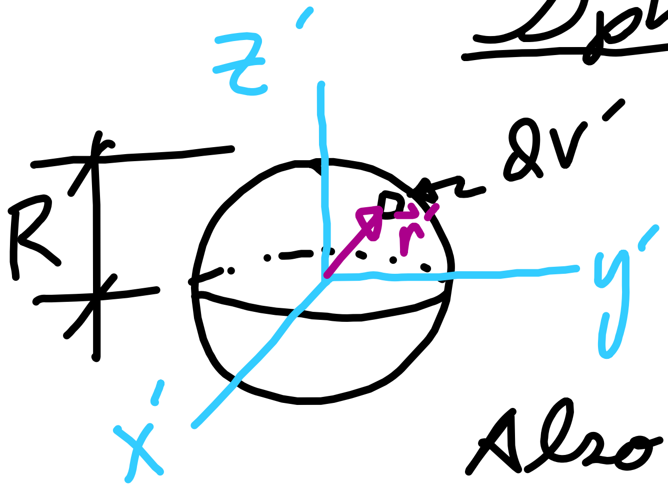

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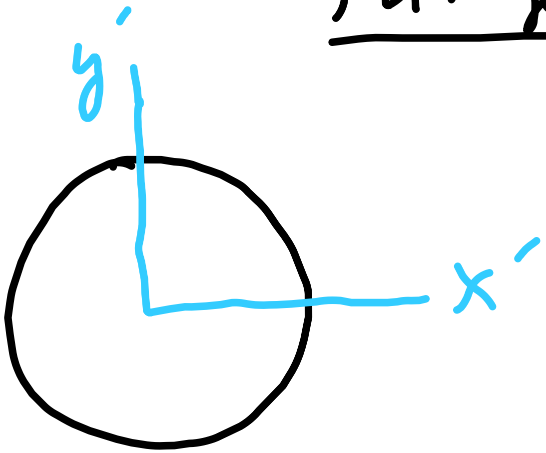
$$\Rightarrow \boxed{I = \frac{2}{5}MR^2}$$

Thin pipe or hoop

Thin pipe or hoop



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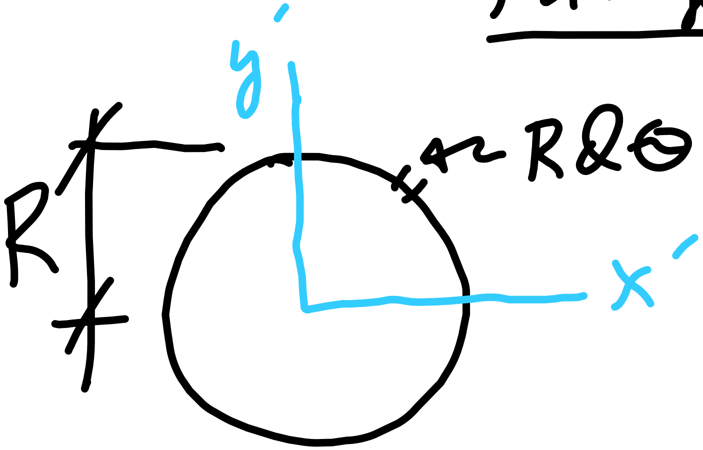
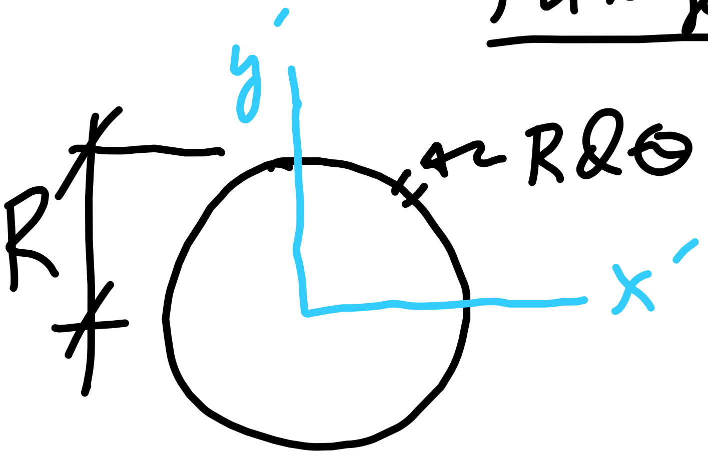


Just need 1 d

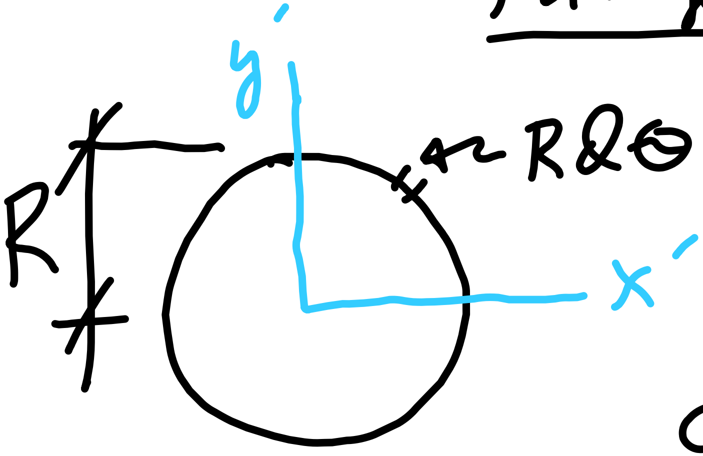
Thin pipe or hoop



Just need 1d

$$\xi \quad \frac{M}{A} \to \frac{M}{C}$$

Thin pipe or hoop

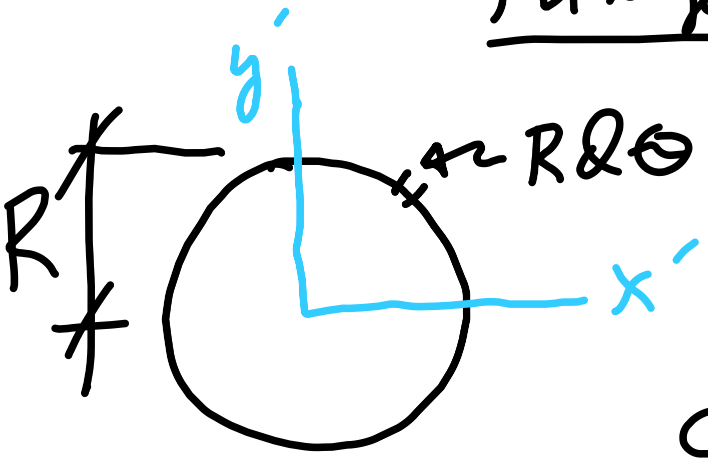


Just need 1d

$\cancel{\frac{M}{A}} + \frac{M}{C}$, where

$C = 2\pi R$

Thin pipe or hoop



Just need 1d

$\oint \quad \frac{M}{A} \rightarrow \frac{M}{C}$, where

$C = 2\pi R$ [circumference of circle]

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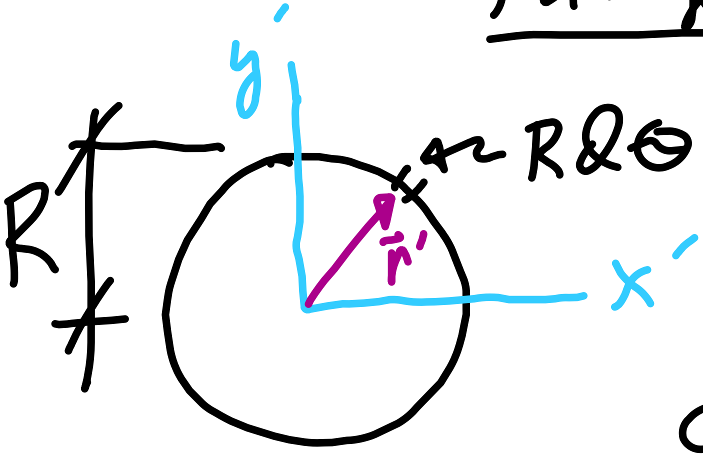

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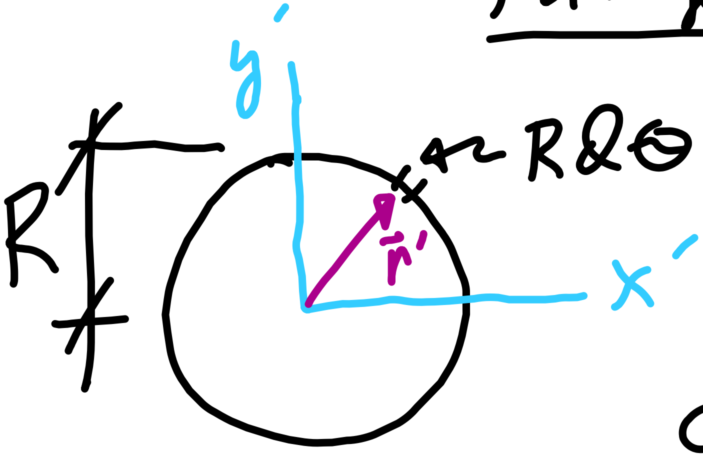

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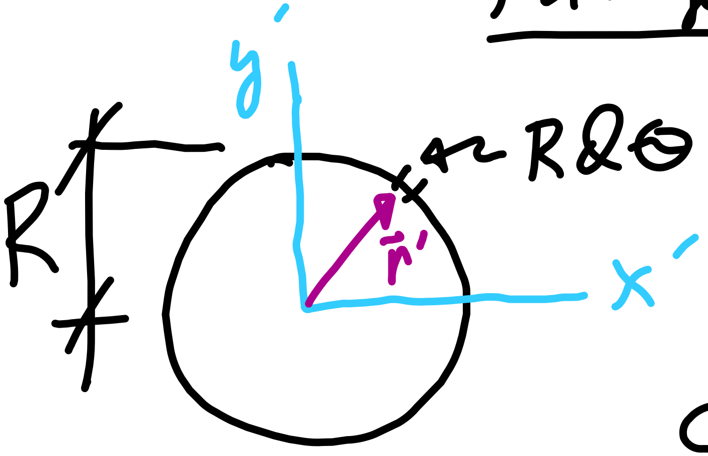
$$\not{\frac{M}{A}} \to \frac{M}{C}, \text{ where}$$

$C = 2\pi R$ [circumference of circle]

Now

$$\bar{I} = \left(\frac{M}{C}\right)\int_0^{2\pi} r'^2 R\, d\theta$$

Thin pipe or hoop



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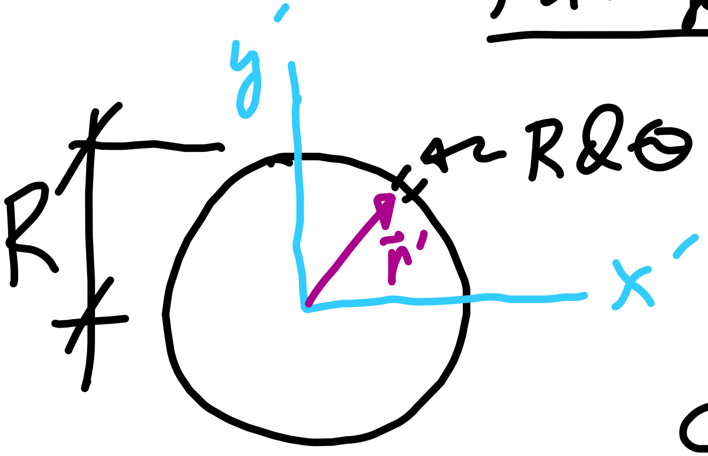
Thin pipe or hoop



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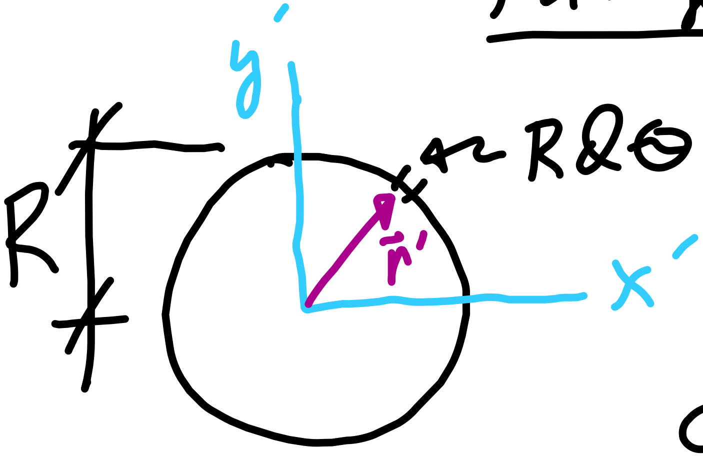
So $I = \frac{M}{C} R^2 \int_0^{2\pi} R \, d\Theta$

Thin pipe or hoop



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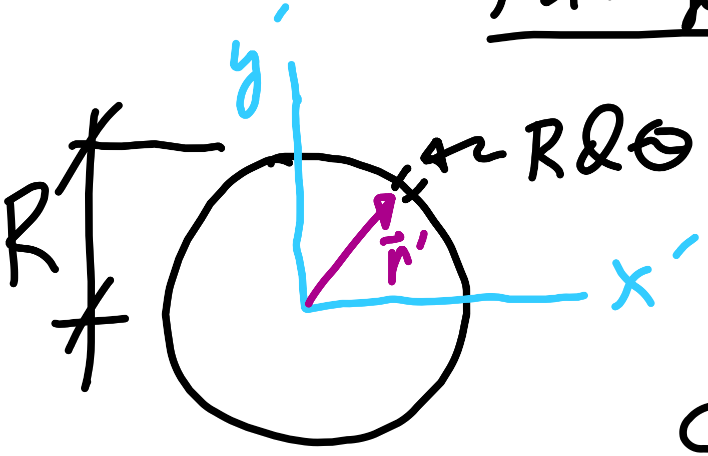

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So $\boxed{I = MR^2}$

Some moments of inertia

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Item	$\bar{I}$
Rectangular plate	$\left(\frac{m}{12}\right)(w^2+L^2)$

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$w \to 0$

Some moments of inertia

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Thin pipe or hoop	MR^2

← all mass located distance R from rotation axis

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Sphere	$\left(\frac{2}{5}\right)MR^2$

For a system of particles

For a system of particles [or rigid body]

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<u>From book</u>: "The system of the external forces and moments is equipollent to the system consisting of the vector $m\vec{\bar{a}}$ attached to G and the couple of moment $\dot{\vec{H}}_G$."

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For plane motion of a rigid body

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$$\Sigma F_x = m\bar{a}_x$$

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For plane motion of a rigid body we have

$$\Sigma F_x = m\bar{a}_x, \quad \Sigma F_y = m\bar{a}_y$$

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What if we wanted to sum moments about a point other than G?

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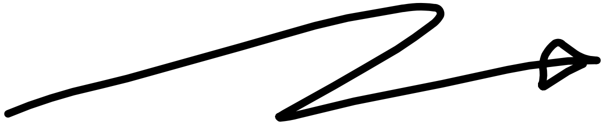
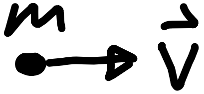
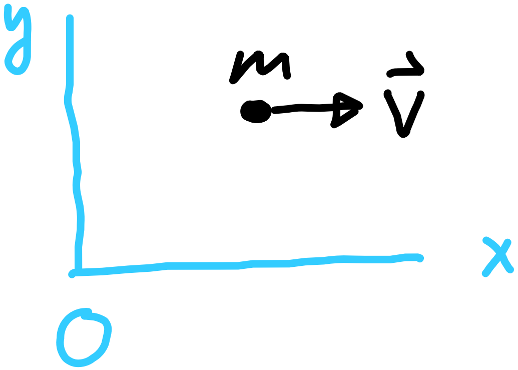
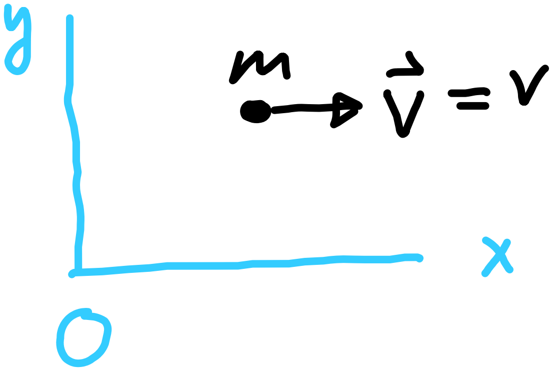
What if we wanted to sum moments about a point other than G?

First we will look at a point particle

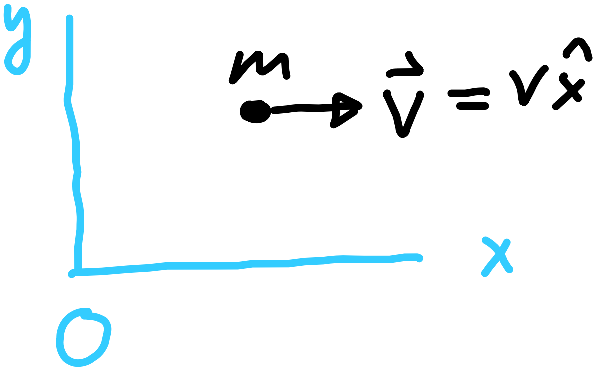



m
$\vec{V}$


y
m
$\vec{V}$
x
O

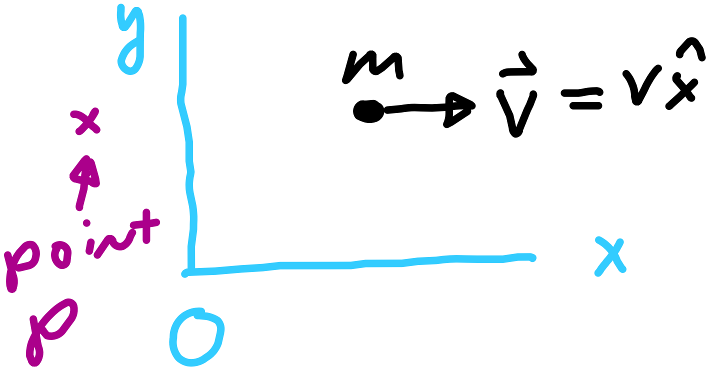


$$\vec{v} = v\hat{x}$$



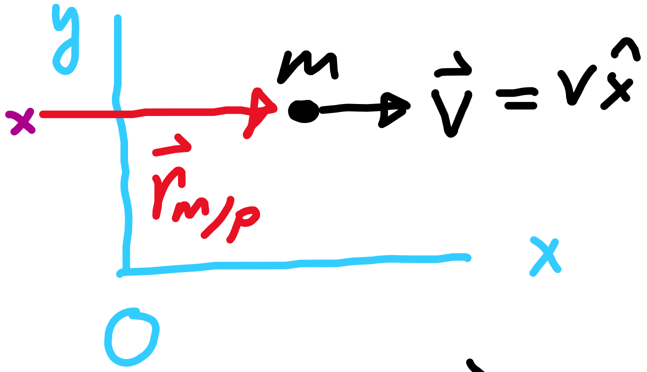
$\vec{v} = v\hat{x}$

$\vec{L} = m\vec{v}$ what about the angular momentum?



$\vec{L} = m\vec{v}$ what about the angular momentum?

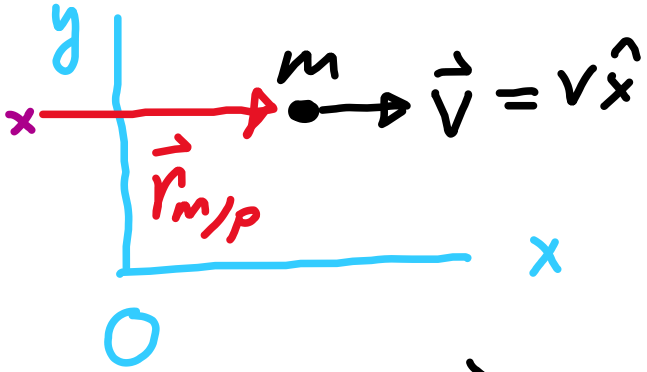
For point p



$\vec{L} = m\vec{v}$ what about the angular momentum?

For point p:

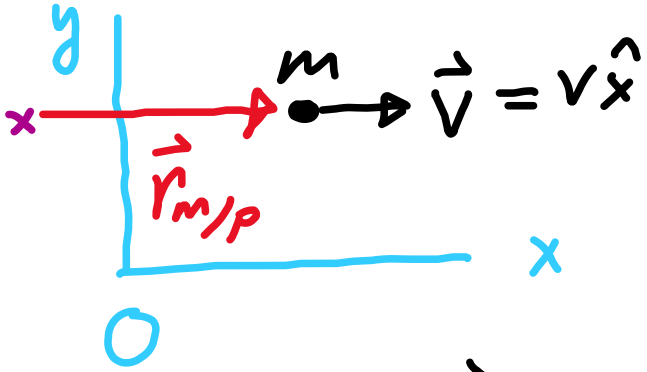
$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L}$$



$\vec{L} = m\vec{v}$ what about the angular momentum?

For point p:

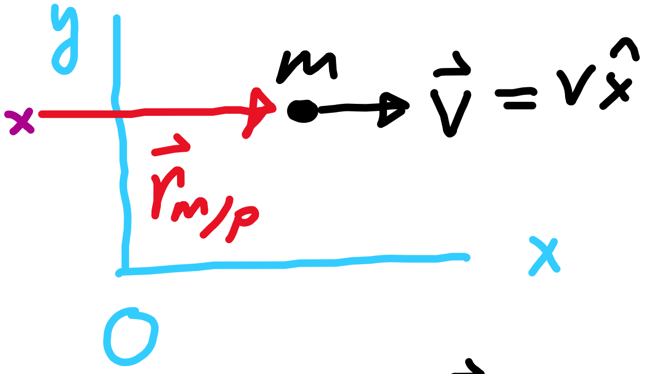
$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x})$$



$\vec{L} = m\vec{v}$ what about the angular momentum?

For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

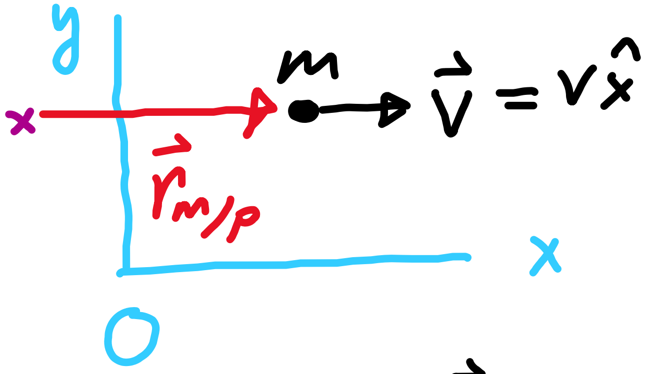


$\vec{L} = m\vec{v}$ what about the angular momentum?

For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

$\Rightarrow$ No angular momentum

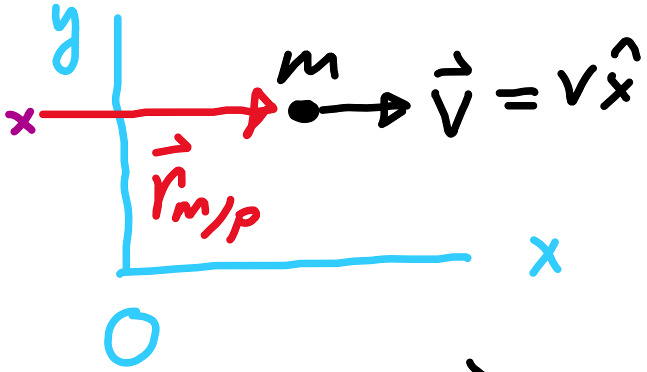


$\vec{L} = m\vec{v}$ what about the angular momentum?

<u>For point p</u>:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

$\Rightarrow$ No angular momentum [as expected].



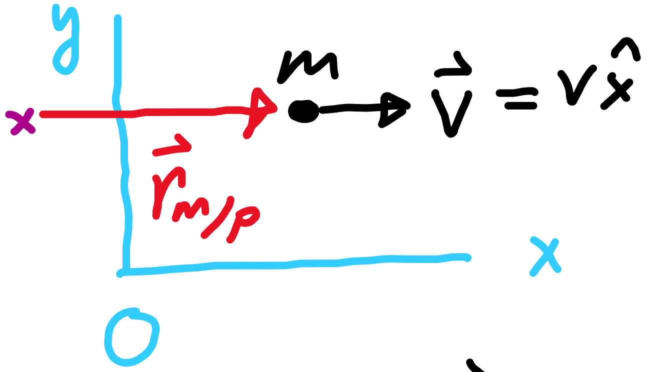
$\vec{L} = m\vec{V}$ what about the angular momentum?

For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mV\hat{x}) = 0$$

$\Rightarrow$ No angular momentum [as expected].

What about $\vec{H}_o$?



$\vec{L} = m\vec{v}$ what about the angular momentum?

For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

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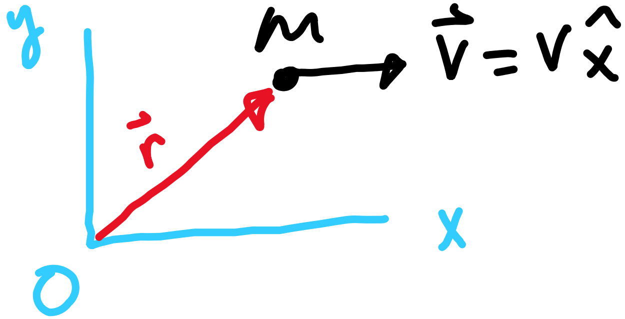
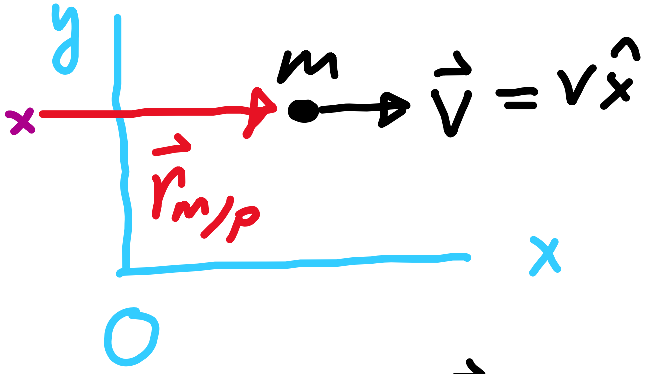
What about $\vec{H}_o$?





$\vec{L} = m\vec{v}$ what about the angular momentum?

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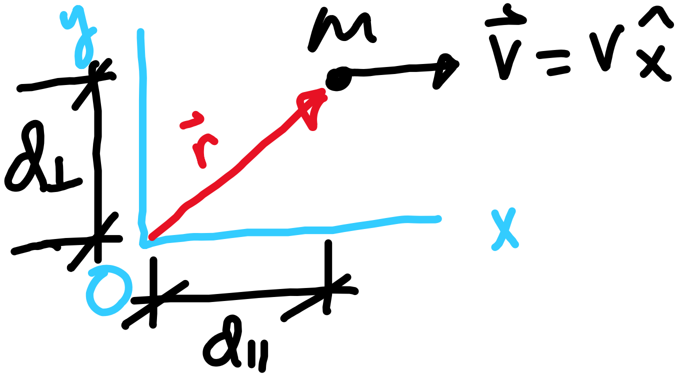
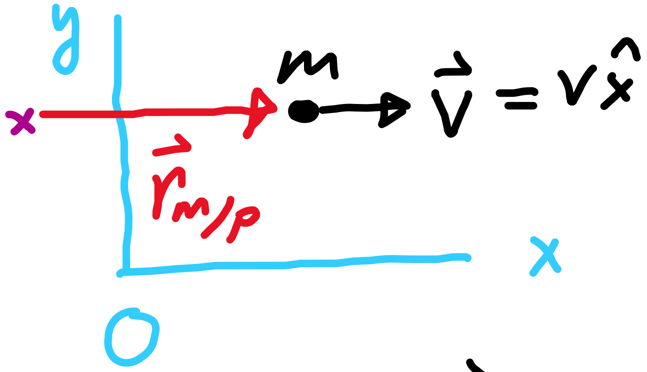
$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

[as expected].

⇒ No angular momentum

What about $\vec{H}_O$?





$\vec{L} = m\vec{v}$ what about the angular momentum?

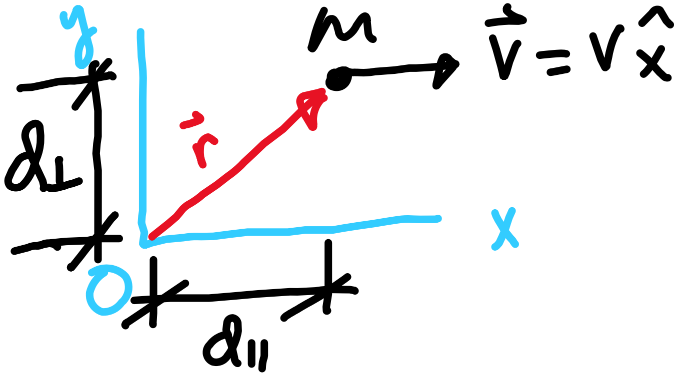
For point p:

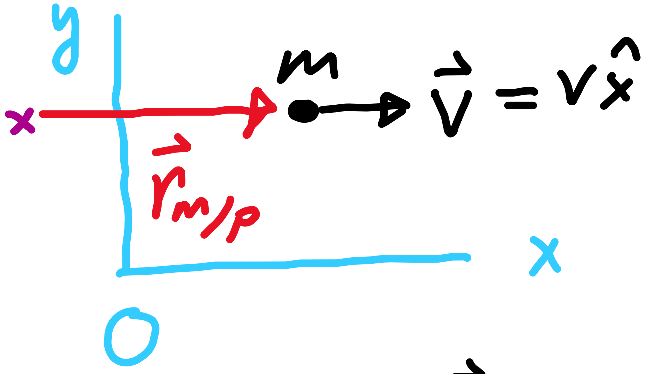
$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mV\hat{x}) = 0$$

[as expected].

$\Rightarrow$ No angular momentum

What about $\vec{H}_o$? As before $\vec{L} = mV\hat{x}$





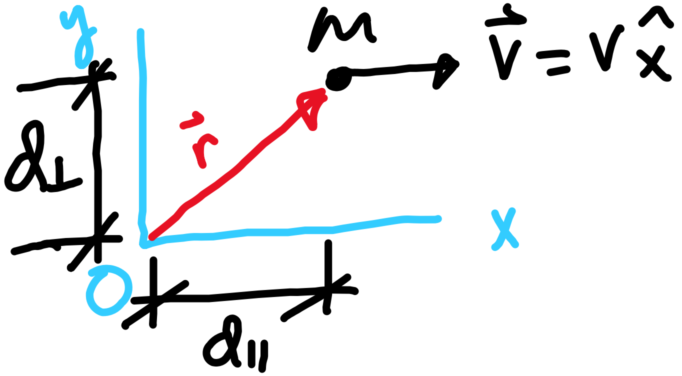
$\vec{L} = m\vec{v}$ what about the angular momentum?

For point p:

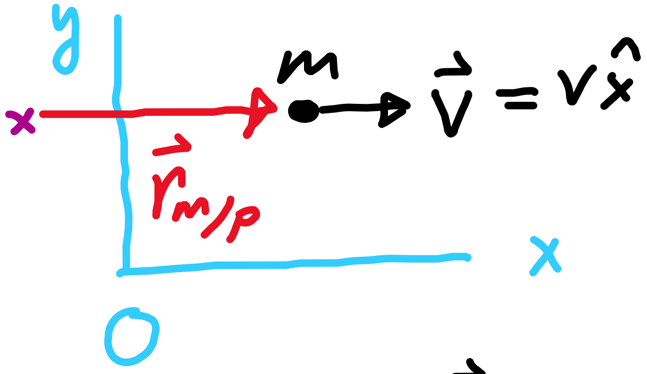
$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

$\Rightarrow$ No angular momentum [as expected].

What about $\vec{H}_o$? As before $\vec{L} = mV\hat{x}$



But now $\vec{H}_o = \vec{r} \times \vec{L}$



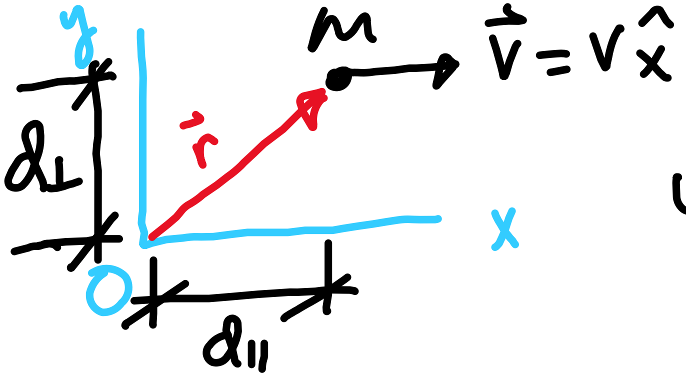
$\vec{L}=m\vec{v}$ what about the angular momentum?

For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

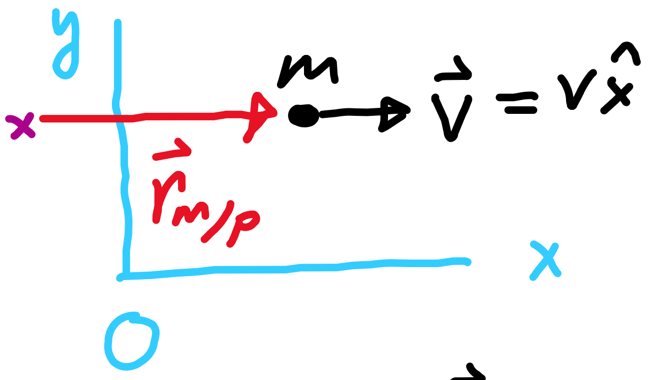
⇒ No angular momentum [as expected].

What about $\vec{H}_o$? As before $\vec{L}=mv\hat{x}$



But now $\vec{H}_o = \vec{r}\times\vec{L}$

with $\vec{r} = d_\parallel \hat{x} + d_\perp \hat{y}$



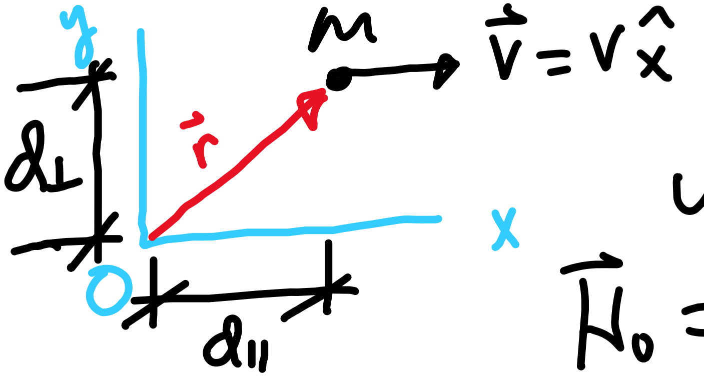
$\vec{L} = m\vec{v}$ what about the angular momentum?

For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

⇒ No angular momentum [as expected].

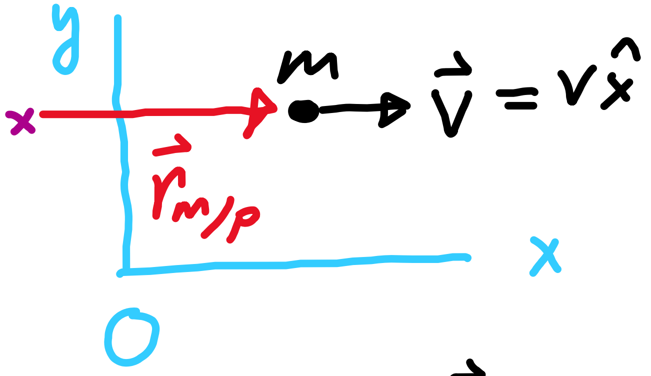
What about $\vec{H}_o$? As before $\vec{L} = mv\hat{x}$



But now $\vec{H}_o = \vec{r} \times \vec{L}$

with $\vec{r} = d_\parallel\hat{x} + d_\perp\hat{y}$ So

$$\vec{H}_o = (d_\parallel\hat{x}) \times (mv\hat{x}) +$$



$\vec{L} = m\vec{v}$ what about the angular momentum?

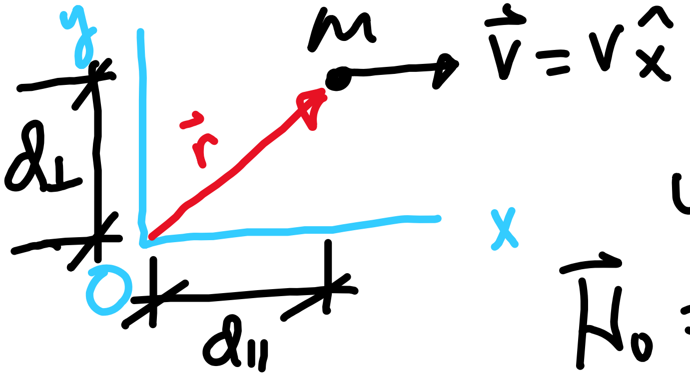
For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

[as expected].

$\Rightarrow$ No angular momentum

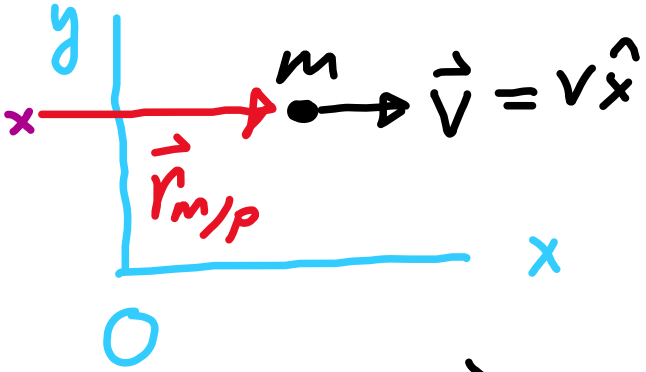
What about $\vec{H}_o$? As before $\vec{L} = mv\hat{x}$



But now $\vec{H}_o = \vec{r} \times \vec{L}$

with $\vec{r} = d_\parallel \hat{x} + d_\perp \hat{y}$ So

$$\vec{H}_o = (d_\parallel \hat{x}) \times (mv\hat{x}) + (d_\perp \hat{y}) \times (mv\hat{y})$$



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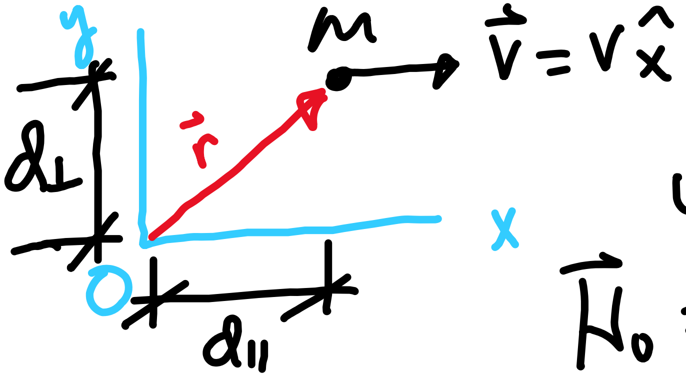

But now $\vec{H}_o = \vec{r} \times \vec{L}$

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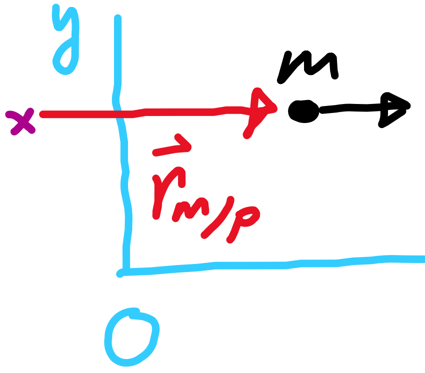
$$\vec{H}_o = \overset{0}{\cancel{(d_\parallel \hat{x}) \times (mv\hat{x})}} + (d_\perp \hat{y}) \times (mv\hat{y})$$



$\vec{v} = v\hat{x}$

$\vec{L} = m\vec{v}$ what about the angular momentum?

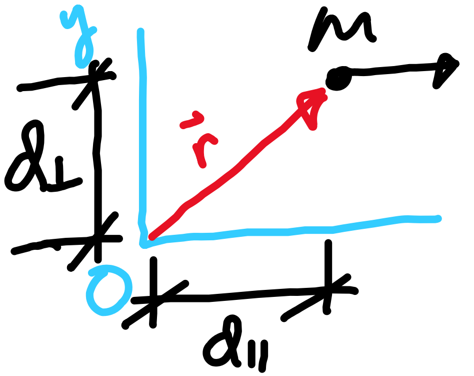
For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

[as expected].

⇒ No angular momentum

What about $\vec{H}_o$? As before $\vec{L} = mv\hat{x}$



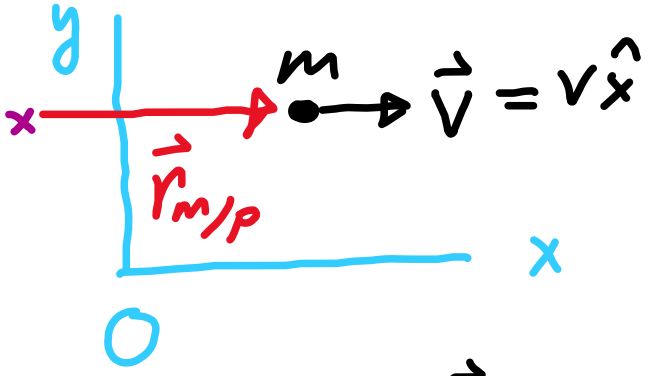
$\vec{v} = v\hat{x}$

But now $\vec{H}_o = \vec{r} \times \vec{L}$

with $\vec{r} = d_\parallel \hat{x} + d_\perp \hat{y}$ So

$$\vec{H}_o = \cancelto{0}{(d_\parallel \hat{x}) \times (mv\hat{x})} + (d_\perp \hat{y}) \times (mv\hat{y})$$

$$\Rightarrow \vec{H}_o = d_\perp m v(-\hat{z})$$



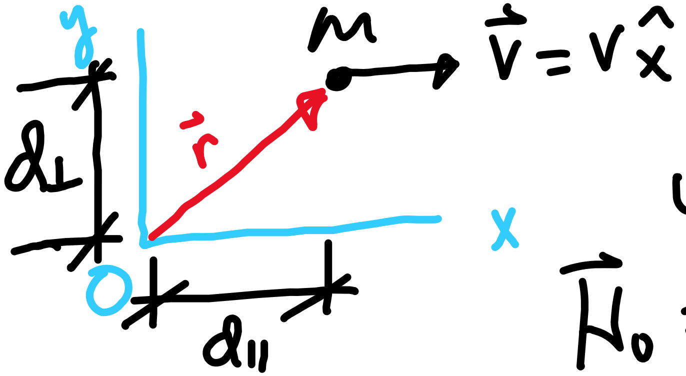
$\vec{L} = m\vec{v}$ what about the angular momentum?

For point p:

$$\vec{H}_p = \vec{r}_{m/p} \times \vec{L} = (r_{m/p}\hat{x}) \times (mv\hat{x}) = 0$$

⇒ No angular momentum [as expected].

What about $\vec{H}_o$? As before $\vec{L} = mv\hat{x}$



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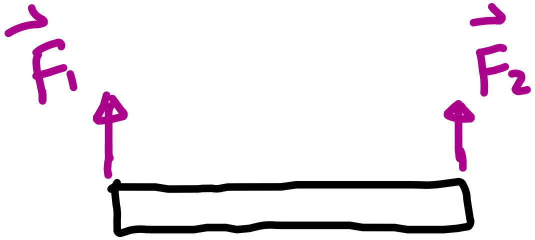
$$\vec{H}_o = \cancelto{0}{(d_\parallel\hat{x}) \times (mv\hat{x})} + (d_\perp\hat{y}) \times (mv\hat{y})$$

$$\Rightarrow \quad \vec{H}_o = d_\perp m v(-\hat{z})$$

In this case $\vec{H}$ is <u>not</u> zero

Slender rod with some forces

Slender rod with some forces



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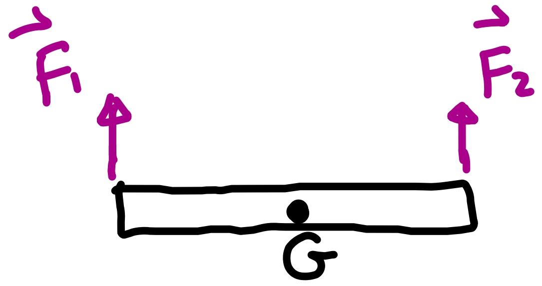

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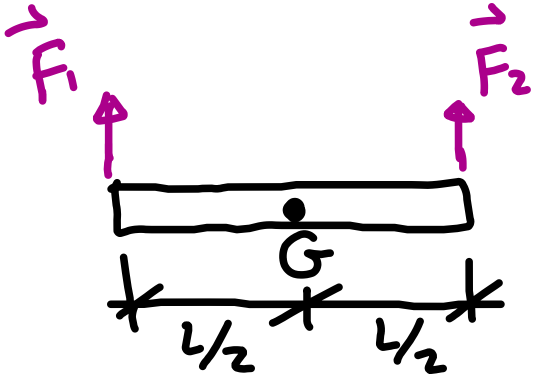

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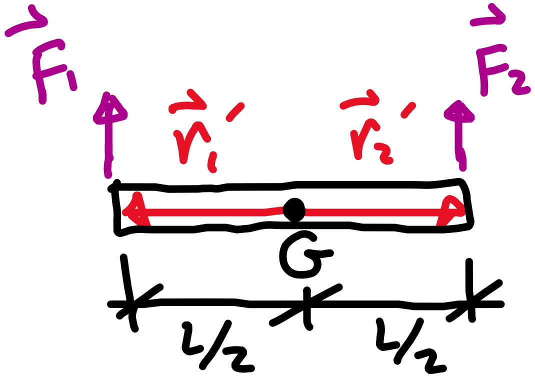
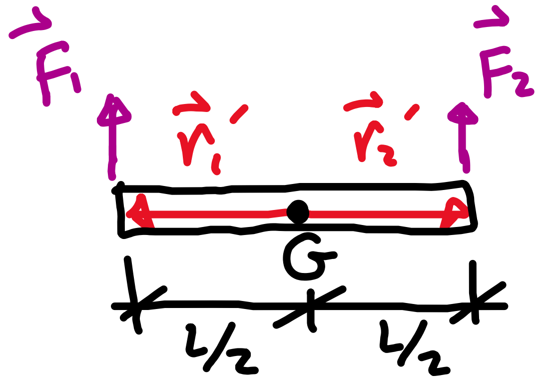
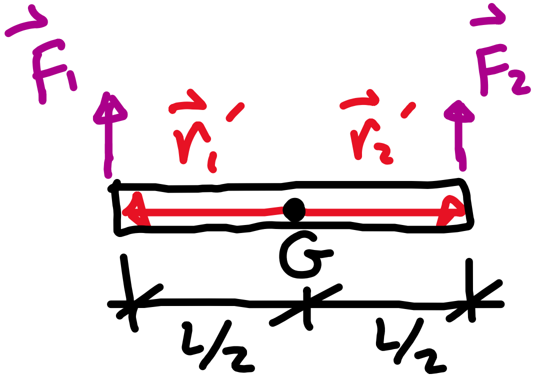


Slender rod with some forces



$$\Sigma \vec{F} = m\vec{\bar{a}}$$

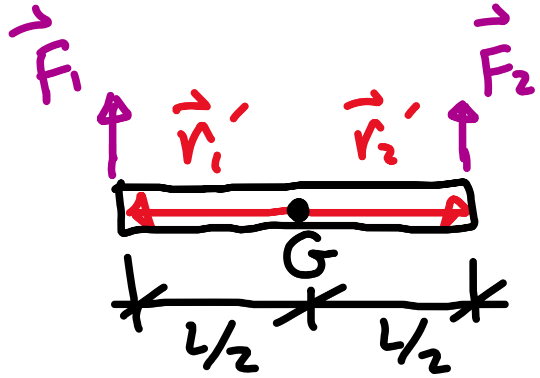
Slender rod with some forces



$$\sum \vec{F} = m\vec{\bar{a}} \ \$$$

$$\sum \vec{M}_G = \vec{r}_1' \times \vec{F}_1 +$$

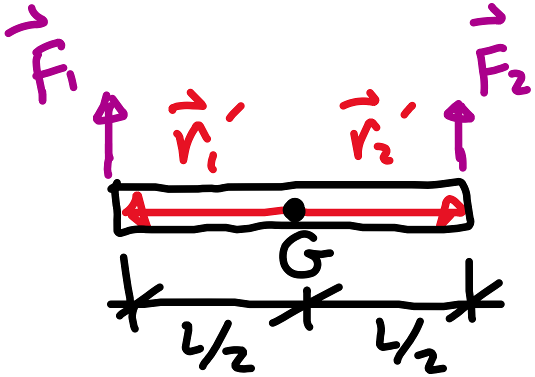
Slender rod with some forces



$$\sum \vec{F} = m\bar{\vec{a}} \;\&$$

$$\sum \vec{M}_G = \vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2$$

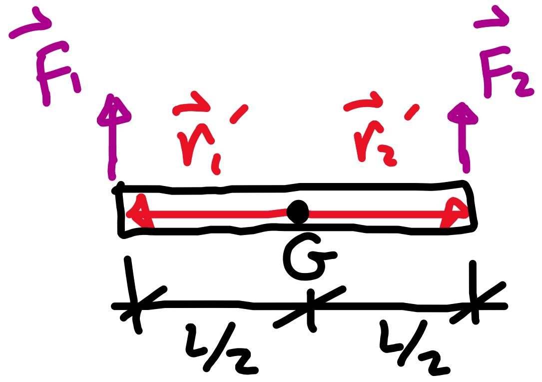
Slender rod with some forces



$$\Sigma \vec{F} = m\vec{\bar{a}} \;\&$$

$$\Sigma \vec{M}_G = \vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2 = \bar{I}\vec{\alpha}$$

Slender rod with some forces



$$\Sigma \vec{F} = m\vec{\bar{a}} \ \&$$

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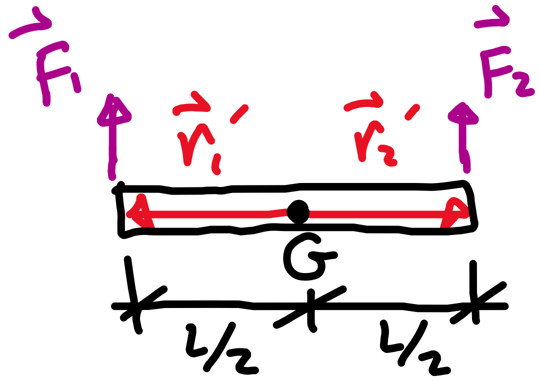
Slender rod with some forces



$$\sum \vec{F} = m\vec{\bar{a}} \;\&$$

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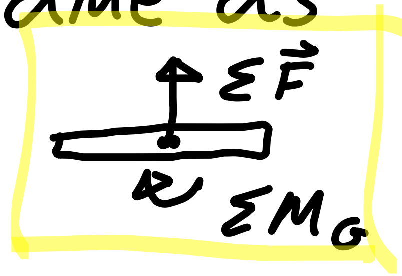
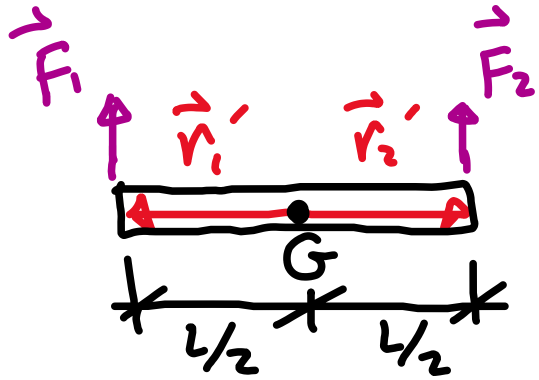


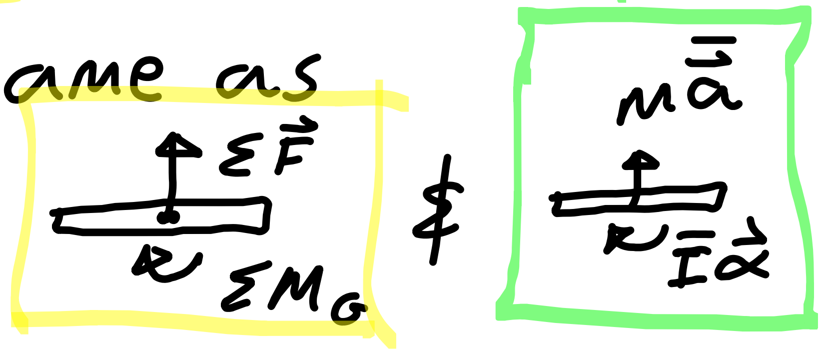
Slender rod with some forces



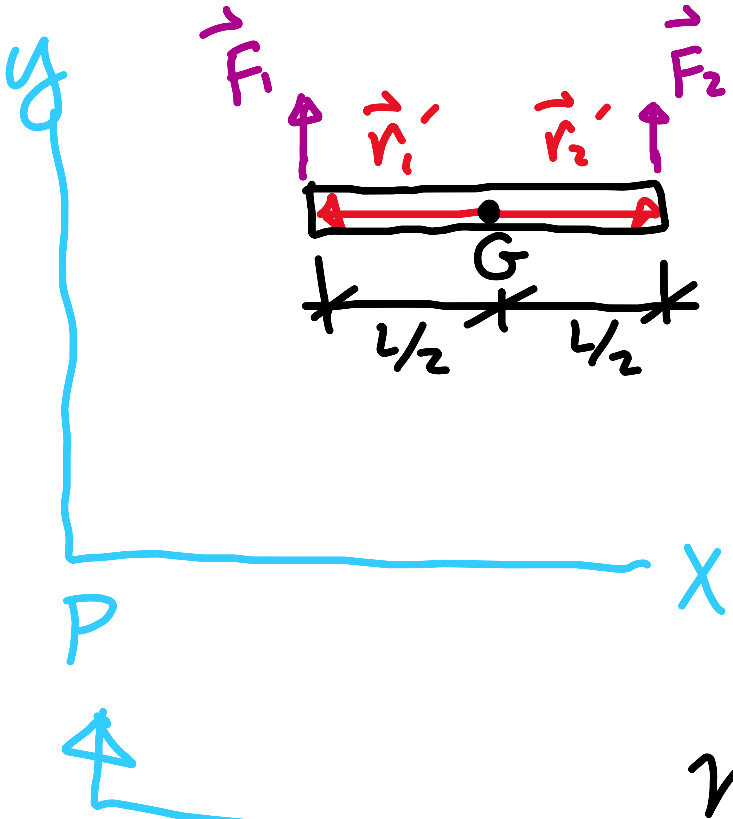
$$\Sigma \vec{F} = m\bar{\vec{a}} \; \&$$

$$\Sigma \vec{M}_G = \vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2 = \bar{I}\vec{\alpha}$$

& same as



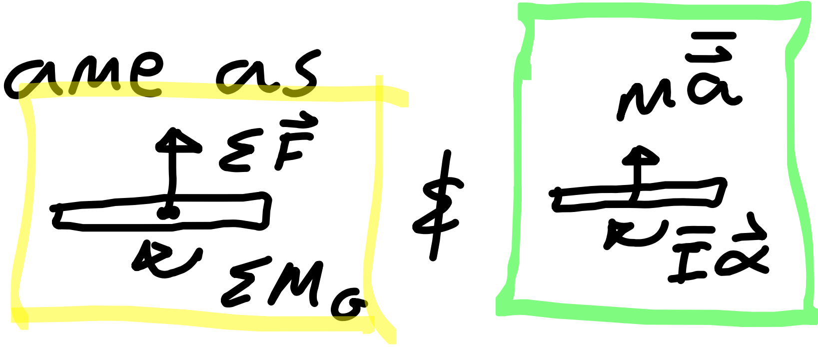
Slender rod with some forces



$$\sum \vec F = m\bar{\vec a} \;\&$$

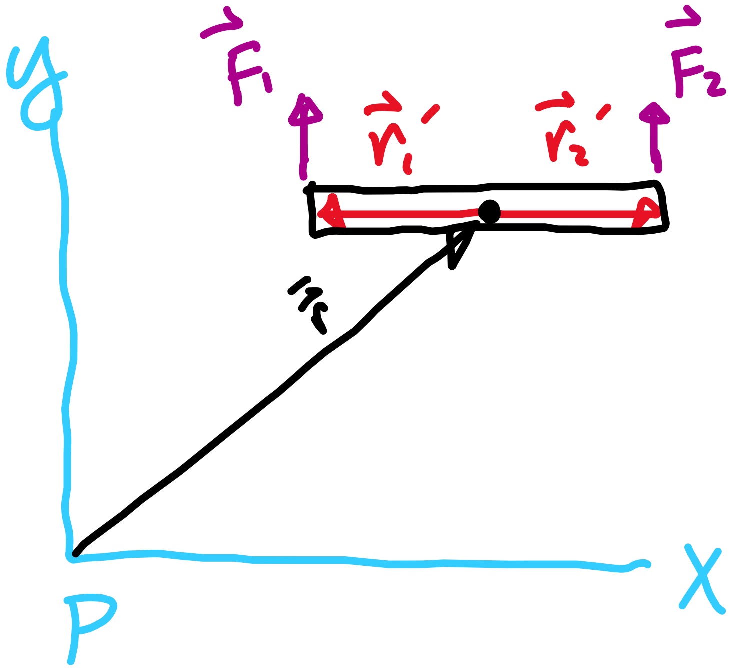
$$\sum \vec M_G = \vec r_1' \times \vec F_1 + \vec r_2 \times \vec F_2 = \bar I \vec \alpha$$

& same as

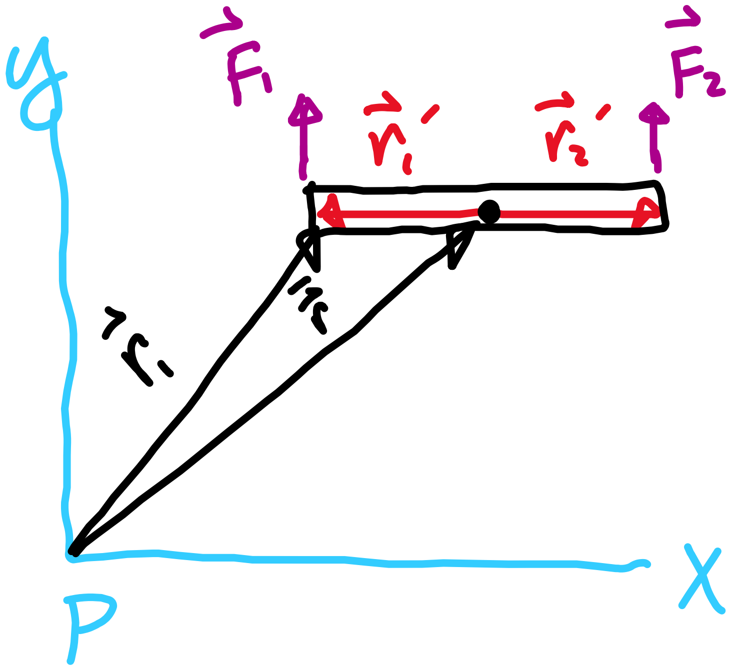


What if we want moments about P?

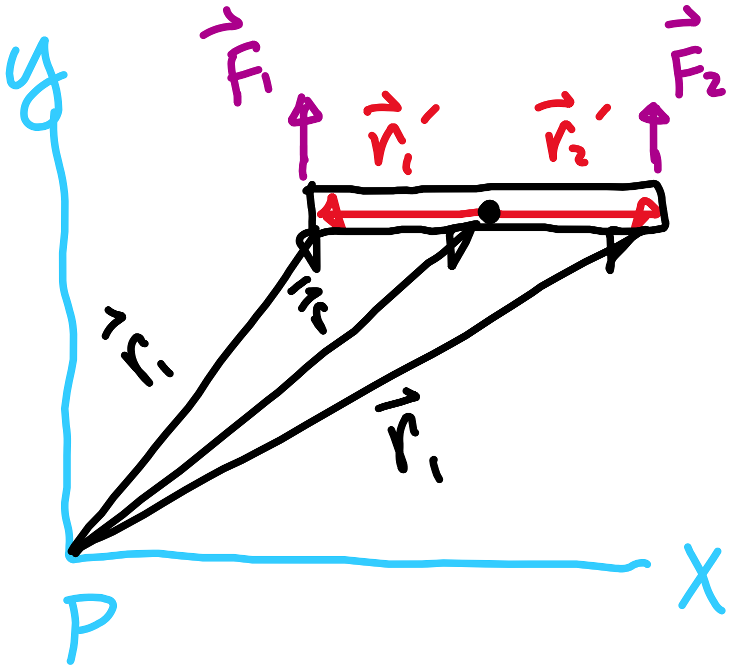
Slender rod with some forces



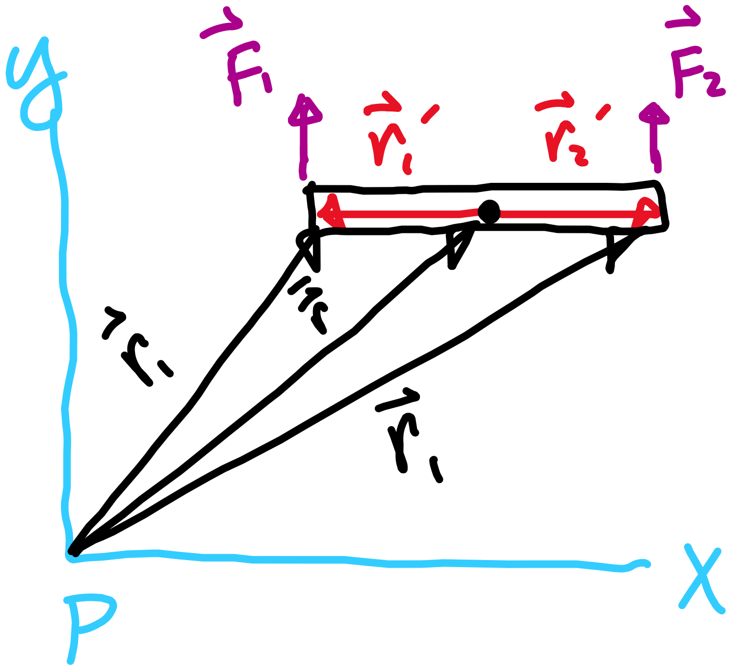
Slender rod with some forces



Slender rod with some forces

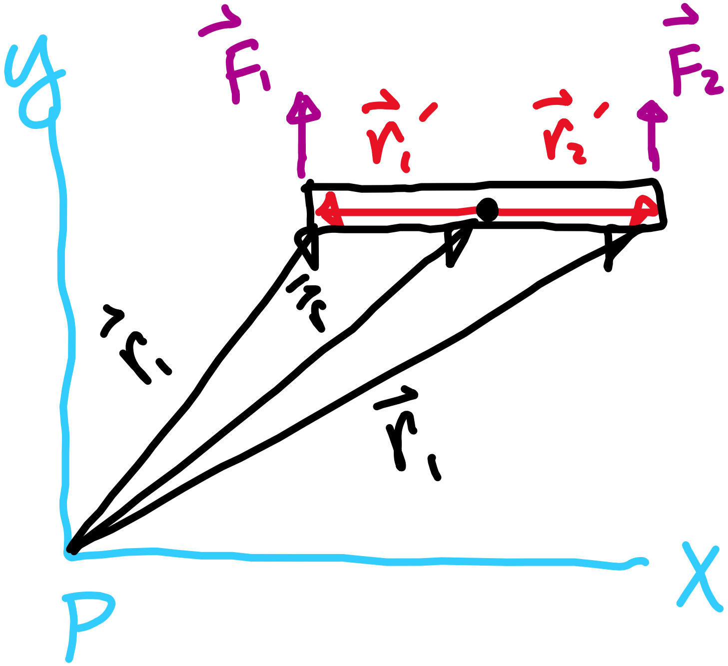


Slender rod with some forces



Note: $\vec{r}_1 = \bar{\vec{r}} + \vec{r}_1'$

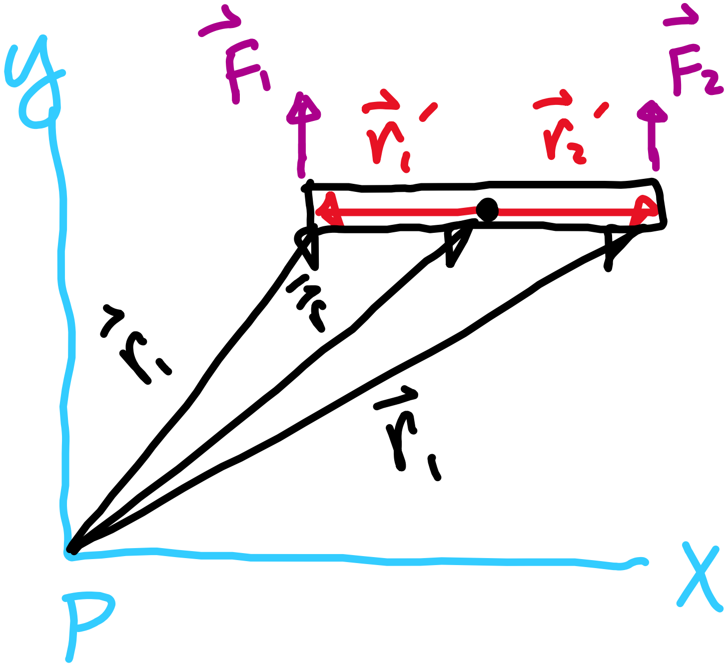
Slender rod with some forces



Note: $\vec{r}_1 = \vec{\bar{r}} + \vec{r}_1{}'$ &

$\vec{r}_2 = \vec{\bar{r}} + \vec{r}_2{}'$

Slender rod with some forces

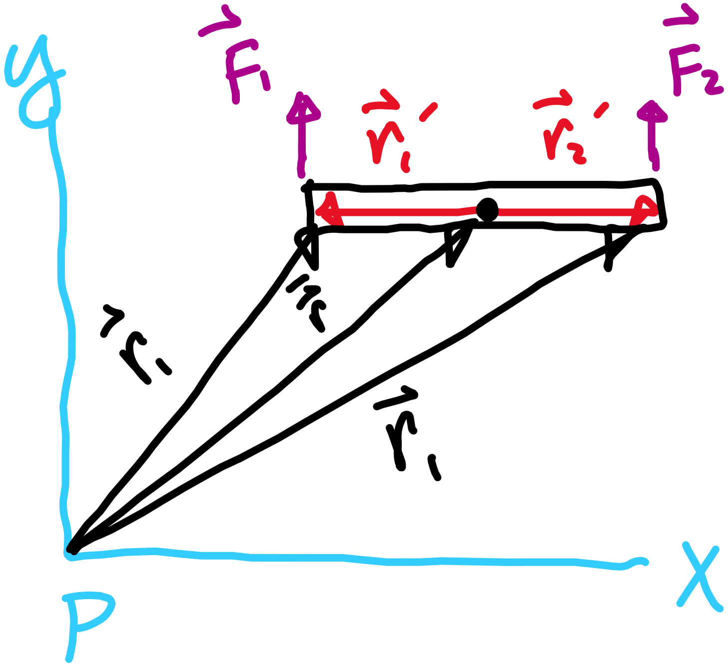


Note: $\vec{r}_1 = \vec{\bar{r}} + \vec{r}_1{}'$ &

$\vec{r}_2 = \vec{\bar{r}} + \vec{r}_2{}'$

Now $\Sigma \vec{M}_P = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$

Slender rod with some forces



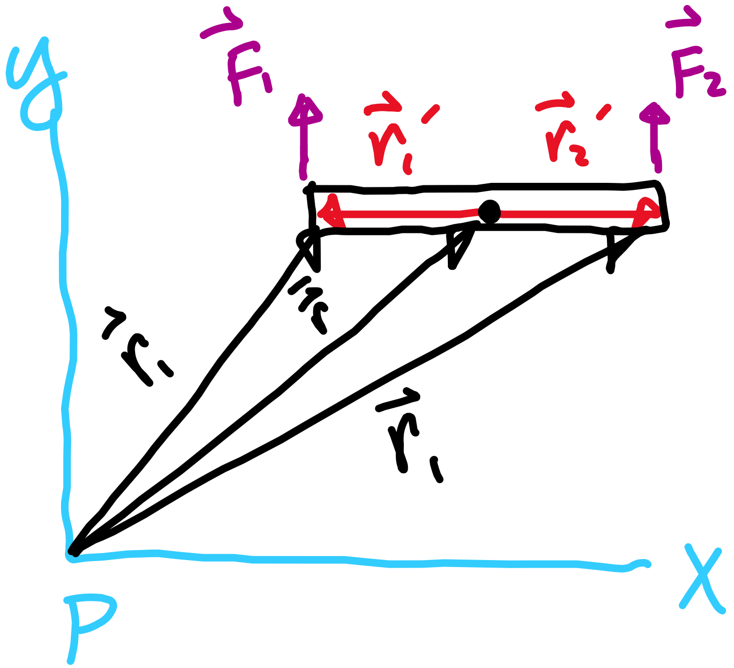
Note: $\vec{r}_1 = \vec{\bar{r}} + \vec{r}_1{}'$ &

$\vec{r}_2 = \vec{\bar{r}} + \vec{r}_2{}'$

Now $\sum \vec{M}_P = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$

$= \vec{\bar{r}} \times \vec{F}_1 + \vec{r}_1{}' \times \vec{F}_1 + \vec{\bar{r}} \times \vec{F}_2 + \vec{r}_2{}' \times \vec{F}_2$

Slender rod with some forces



Note: $\vec r_1 = \vec{\bar r} + \vec r_1'$ &

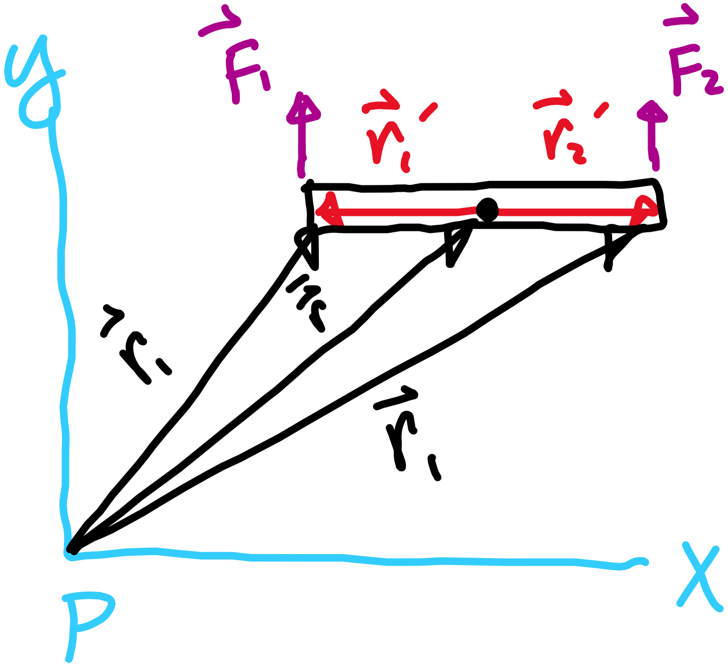
$\vec r_2 = \vec{\bar r} + \vec r_2'$

Now $\sum \vec M_P = \vec r_1 \times \vec F_1 + \vec r_2 \times \vec F_2$

$$= \vec{\bar r} \times \vec F_1 + \vec r_1' \times \vec F_1 + \vec{\bar r} \times \vec F_2 + \vec r_2' \times \vec F_2$$

$$= \vec{\bar r} \times \sum \vec F + [\vec r_1' \times \vec F_1 + \vec r_2' \times \vec F_2]$$

Slender rod with some forces



Note: $\vec{r}_1 = \vec{\bar{r}} + \vec{r}_1'$ &

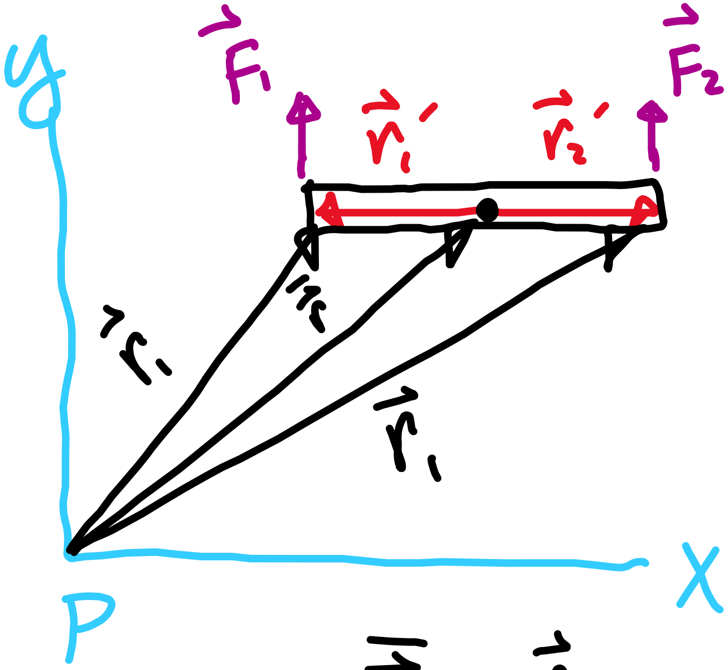
$\vec{r}_2 = \vec{\bar{r}} + \vec{r}_2'$

Now $\Sigma \vec{M}_P = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$

$= \vec{\bar{r}} \times \vec{F}_1 + \vec{r}_1' \times \vec{F}_1 + \vec{\bar{r}} \times \vec{F}_2 + \vec{r}_2' \times \vec{F}_2$

$$= \vec{\bar{r}} \times \Sigma \vec{F} + \underbrace{\left[\vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2\right]}_{\text{Same as before}}$$

Slender rod with some forces



Note: $\vec r_1 = \vec{\bar r} + \vec r_1'$ &

$\vec r_2 = \vec{\bar r} + \vec r_2'$

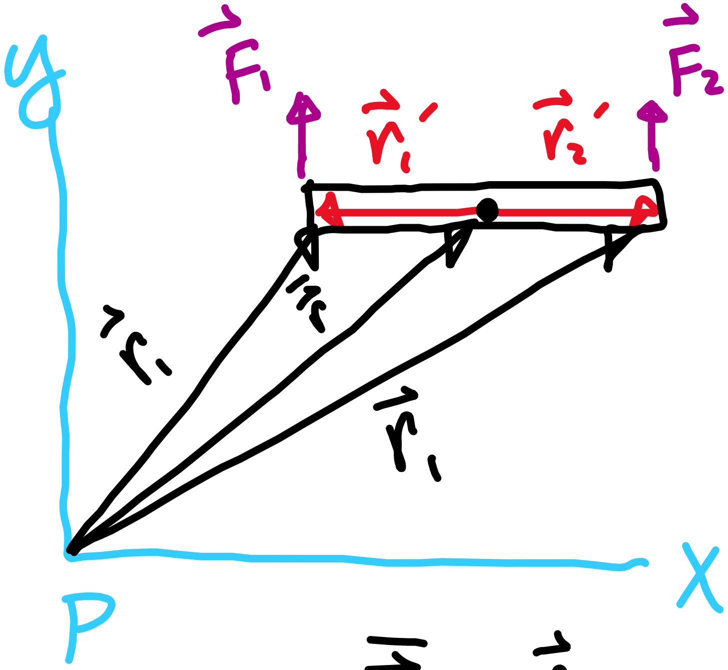
Now $\Sigma \vec M_P = \vec r_1 \times \vec F_1 + \vec r_2 \times \vec F_2$

$= \vec{\bar r} \times \vec F_1 + \vec r_1' \times \vec F_1 + \vec{\bar r} \times \vec F_2 + \vec r_2' \times \vec F_2$

$= \vec{\bar r} \times \Sigma \vec F + \underbrace{[\vec r_1' \times \vec F_1 + \vec r_2' \times \vec F_2]}_{\text{Same as before}}$

Note: $\vec{\bar r} = \vec r_{G/P}$

Slender rod with some forces



Note: $\vec r_1 = \bar{\vec r} + \vec r_1'$ &

$\vec r_2 = \bar{\vec r} + \vec r_2'$

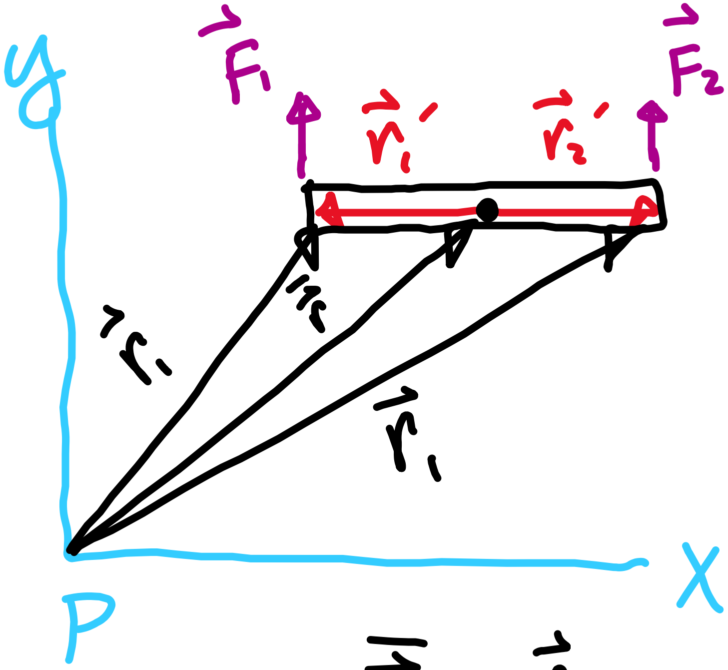
Now $\Sigma \vec M_P = \vec r_1 \times \vec F_1 + \vec r_2 \times \vec F_2$

$$= \bar{\vec r} \times \vec F_1 + \vec r_1' \times \vec F_1 + \bar{\vec r} \times \vec F_2 + \vec r_2' \times \vec F_2$$

$$= \bar{\vec r} \times \Sigma \vec F + \underbrace{[\vec r_1' \times \vec F_1 + \vec r_2' \times \vec F_2]}_{\text{Same as before}}$$

Note: $\bar{\vec r} = \vec r_{G/P}$ so $\vec M_P = \vec r_{G/P} \times \Sigma \vec F + \vec M_G$

Slender rod with some forces



Note: $\vec{r}_1 = \vec{\bar{r}} + \vec{r}_1'$ &

$\vec{r}_2 = \vec{\bar{r}} + \vec{r}_2'$

Now $\Sigma \vec{M}_P = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$

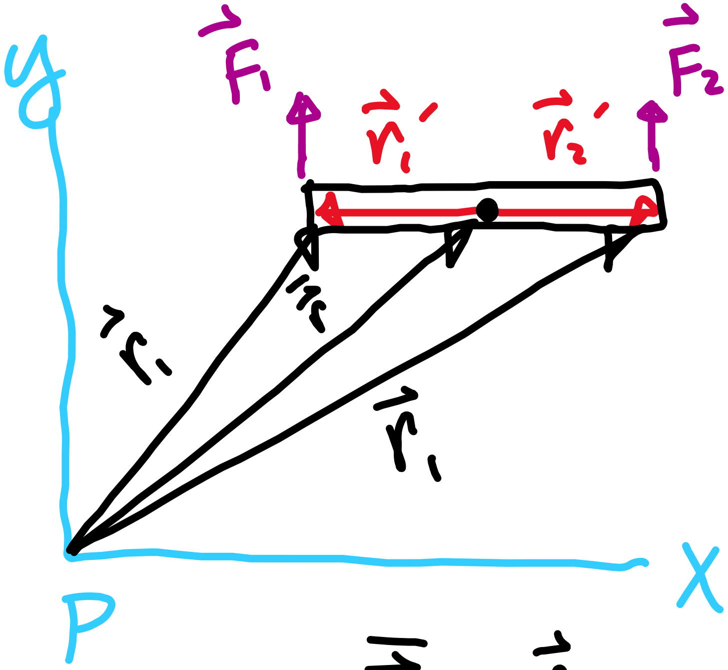
$= \vec{\bar{r}} \times \vec{F}_1 + \vec{r}_1' \times \vec{F}_1 + \vec{\bar{r}} \times \vec{F}_2 + \vec{r}_2' \times \vec{F}_2$

$= \vec{\bar{r}} \times \Sigma \vec{F} + \underbrace{[\vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2]}_{\text{Same as before}}$

Note: $\vec{\bar{r}} = \vec{r}_{G/P}$ so $\vec{M}_P = \vec{r}_{G/P} \times \Sigma \vec{F} + \vec{M}_G$

& since $\Sigma \vec{F} = m\vec{\bar{a}}$

Slender rod with some forces



Note: $\vec{r}_1 = \vec{\bar{r}} + \vec{r}_1'$ &

$\vec{r}_2 = \vec{\bar{r}} + \vec{r}_2'$

Now $\Sigma \vec{M}_P = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$

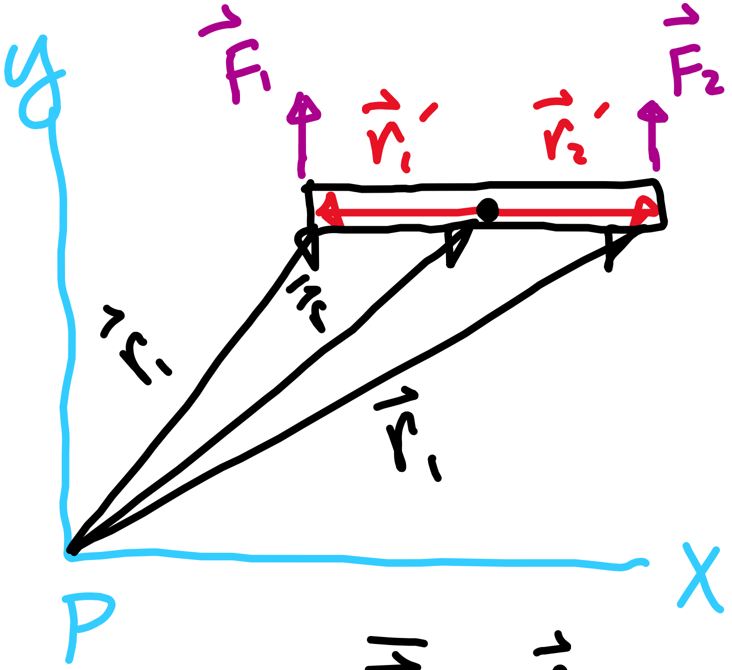
$= \vec{\bar{r}} \times \vec{F}_1 + \vec{r}_1' \times \vec{F}_1 + \vec{\bar{r}} \times \vec{F}_2 + \vec{r}_2' \times \vec{F}_2$

$= \vec{\bar{r}} \times \Sigma \vec{F} + \underbrace{[\vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2]}_{\text{Same as before}}$

Note: $\vec{\bar{r}} = \vec{r}_{G/P}$ so $\vec{M}_P = \vec{r}_{G/P} \times \Sigma \vec{F} + \vec{M}_G$

& since $\Sigma \vec{F} = m\vec{\bar{a}}$ & $\vec{M}_G = \bar{I}\vec{\alpha}$

Slender rod with some forces



Note: $\vec{r}_1 = \vec{\bar{r}} + \vec{r}_1{}'$ &

$\vec{r}_2 = \vec{\bar{r}} + \vec{r}_2{}'$

Now $\Sigma \vec{M}_P = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$

$$= \vec{\bar{r}} \times \vec{F}_1 + \vec{r}_1{}' \times \vec{F}_1 + \vec{\bar{r}} \times \vec{F}_2 + \vec{r}_2{}' \times \vec{F}_2$$

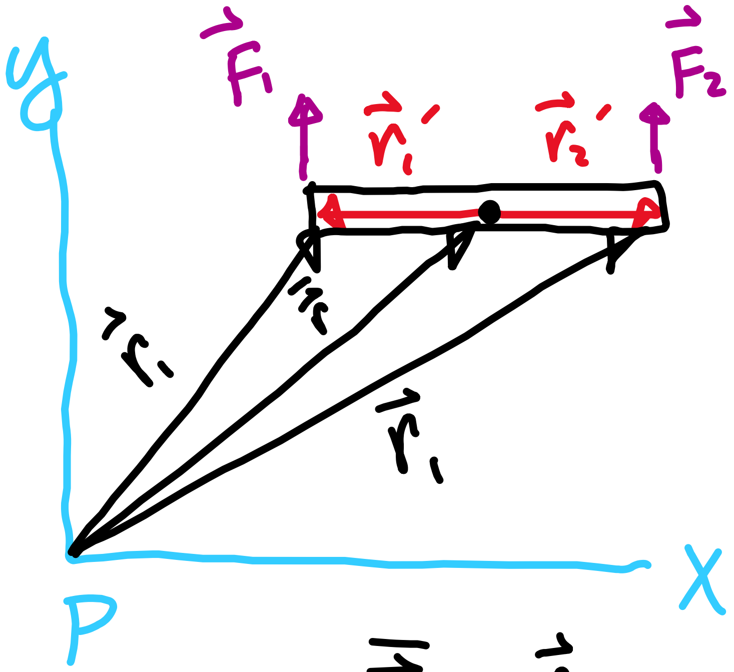
$$= \vec{\bar{r}} \times \Sigma \vec{F} + \underbrace{[\vec{r}_1{}' \times \vec{F}_1 + \vec{r}_2{}' \times \vec{F}_2]}_{\text{Same as before}}$$

Note: $\vec{\bar{r}} = \vec{r}_{G/P}$ so $\vec{M}_P = \vec{r}_{G/P} \times \Sigma \vec{F} + \vec{M}_G$

& since $\Sigma \vec{F} = m\vec{\bar{a}}$ & $\vec{M}_G = \bar{I}\vec{\alpha}$ then

$$\boxed{\vec{M}_P = \underbrace{\vec{r}_{G/P} \times m\vec{\bar{a}}}_{\text{New piece}} + \bar{I}\vec{\alpha}}$$

Slender rod with some forces



Note: $\vec r_1 = \vec{\bar r} + \vec r_1'$ &

$\vec r_2 = \vec{\bar r} + \vec r_2'$

Now $\Sigma \vec M_P = \vec r_1 \times \vec F_1 + \vec r_2 \times \vec F_2$

$= \vec{\bar r} \times \vec F_1 + \vec r_1' \times \vec F_1 + \vec{\bar r} \times \vec F_2 + \vec r_2' \times \vec F_2$

$= \vec{\bar r} \times \Sigma \vec F + \underbrace{[\vec r_1' \times \vec F_1 + \vec r_2' \times \vec F_2]}_{\text{Same as before}}$

Note: $\vec{\bar r} = \vec r_{G/P}$ so $\vec M_P = \vec r_{G/P} \times \Sigma \vec F + \vec M_G$

& since $\Sigma \vec F = m\vec{\bar a}$ & $\vec M_G = \bar I \vec \alpha$ then

$$\boxed{\vec M_P = \underbrace{\vec r_{G/P} \times m\vec{\bar a}}_{\text{New piece}} + \bar I \vec \alpha}$$

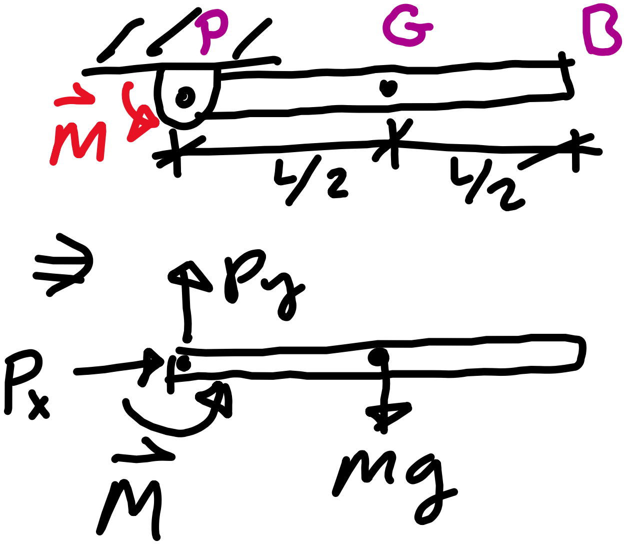
New piece → $\dot{\vec H}_P$ of point particle of mass m at G

Example

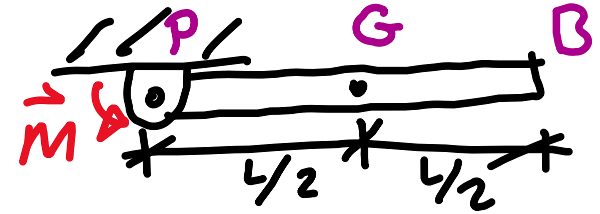
Example

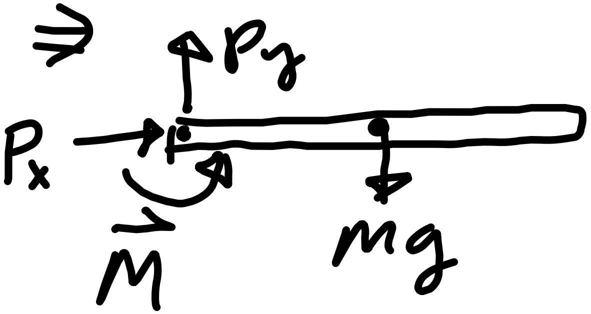
L_x^y

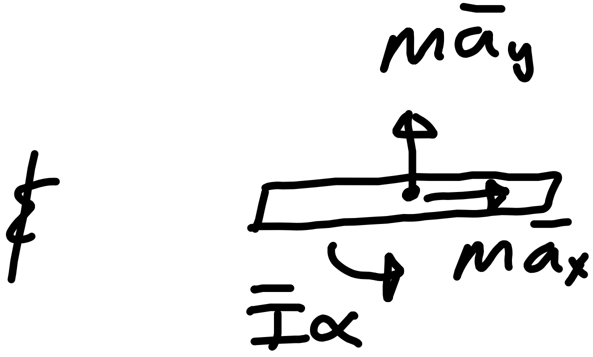
Example



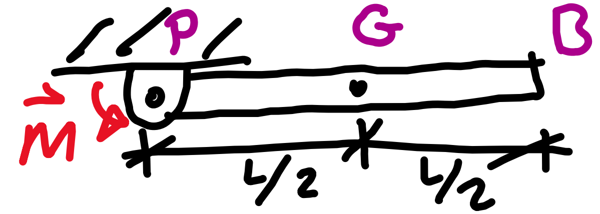
Example

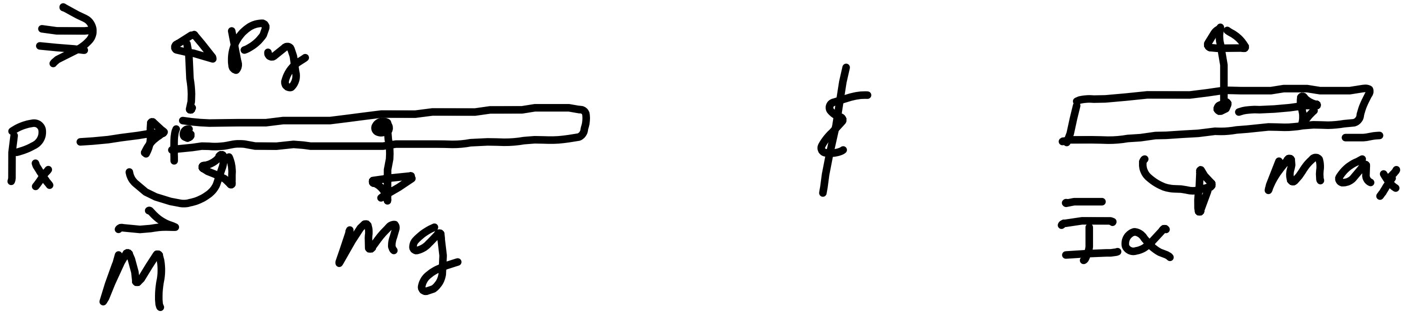






Example

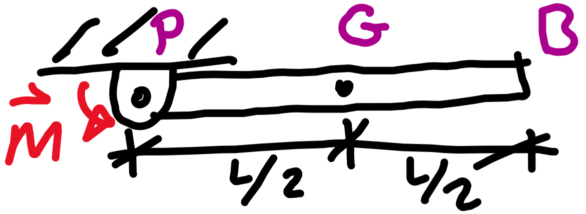


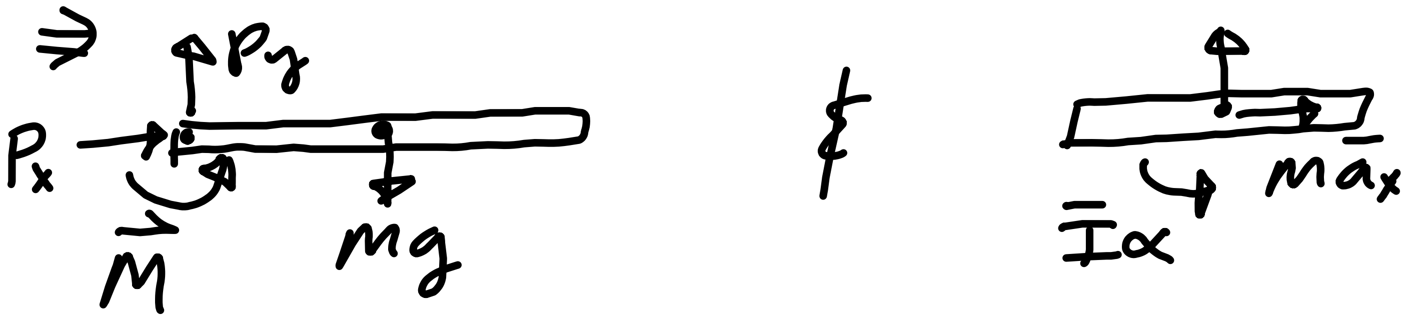


About G: $\circlearrowleft \Sigma M_G = \bar{I}\alpha$

Example

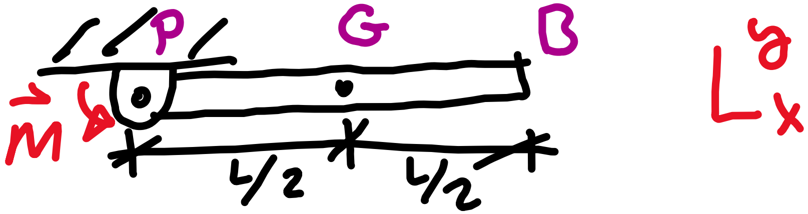
L_x^y

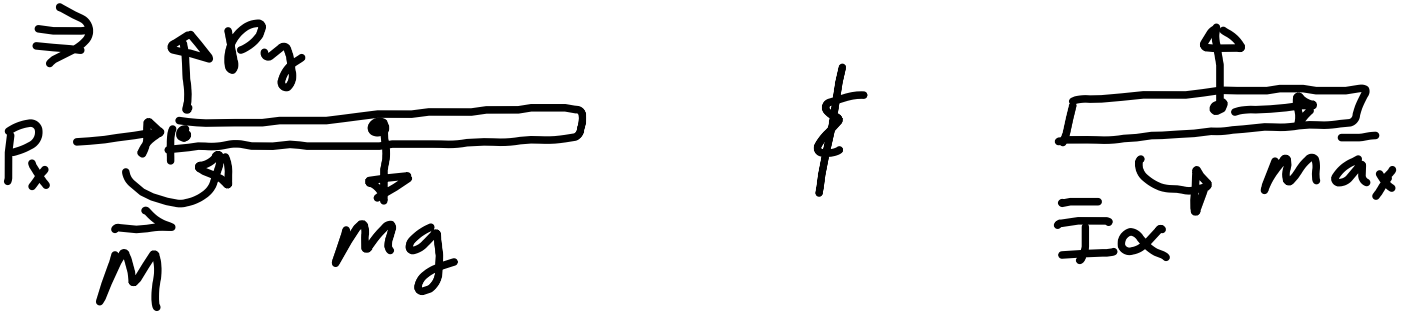




About G: $\circlearrowleft \sum M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2} P_y = \bar{I}\alpha$

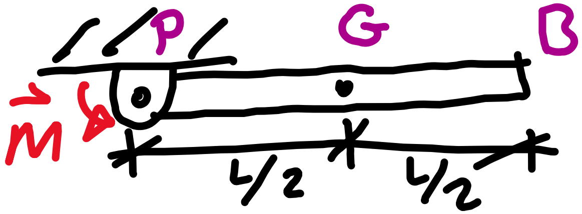
Example

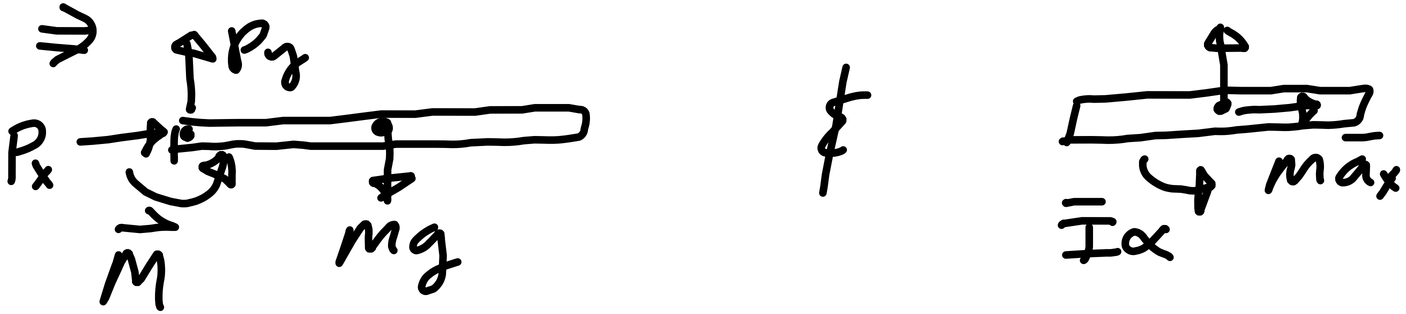




About G: $\circlearrowleft \Sigma M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha \quad (1)$

Example

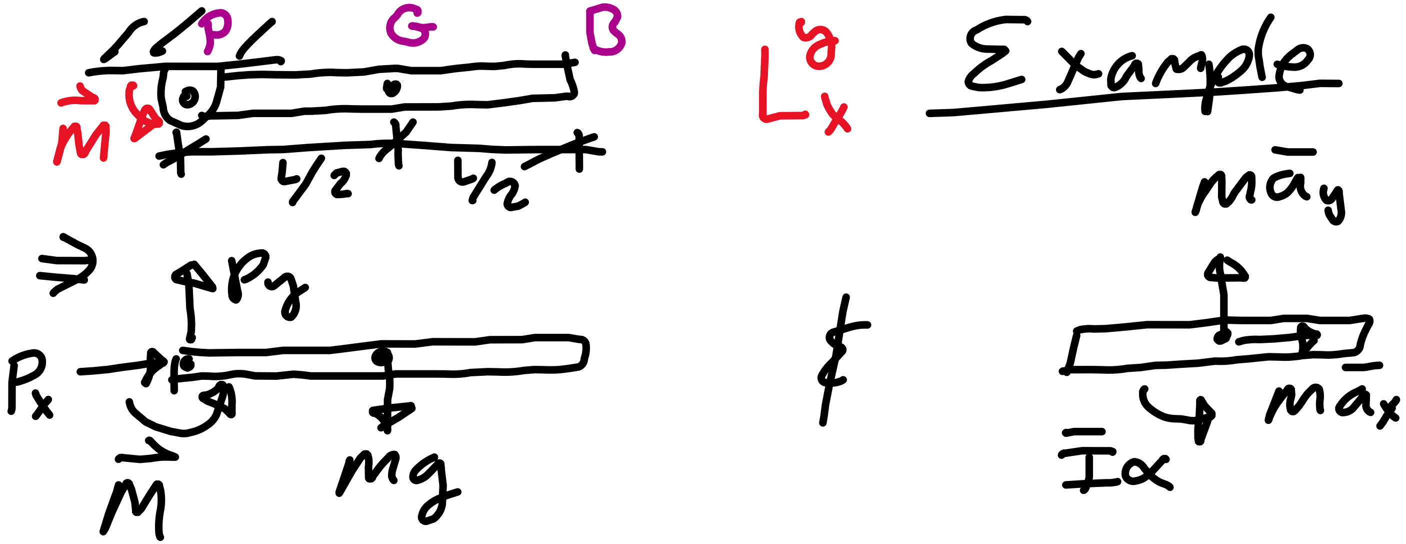




About G: $\circlearrowleft \sum M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha$ (1)

About P: $\circlearrowleft \sum M_P = \bar{I}\alpha + \vec{r}_{G/P} \times m\bar{\bar{a}}$

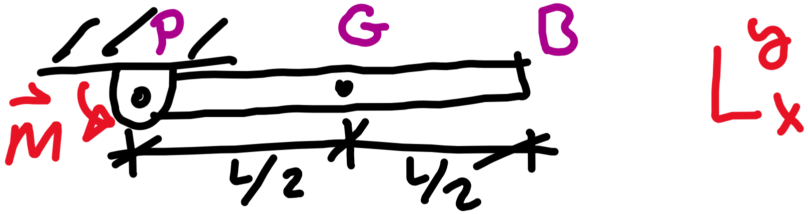
Example

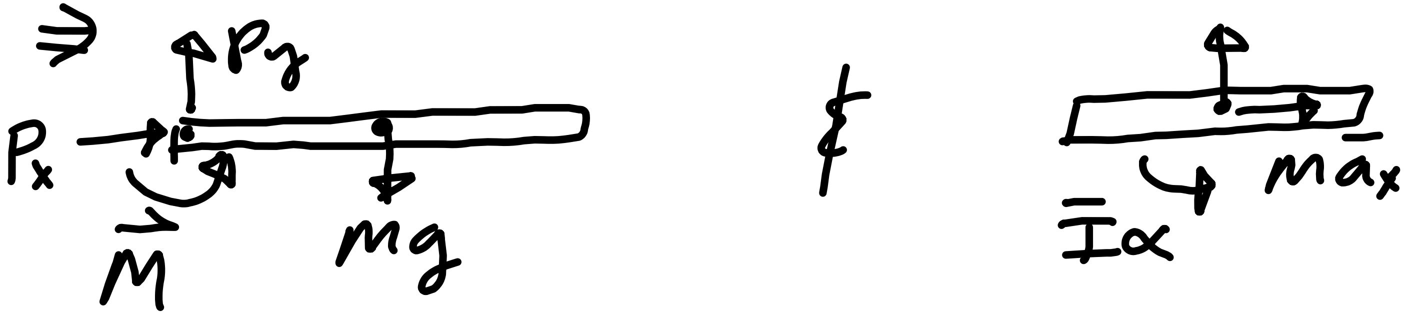


About G: $\circlearrowleft \sum M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha$ (1)

About P: $\circlearrowleft \sum M_P = \bar{I}\alpha + \vec{r}_{G/P} \times m\bar{\vec{a}} \Rightarrow M - \frac{L}{2}mg = \bar{I}\alpha + \frac{mL\bar{a}_y}{2}$

Example



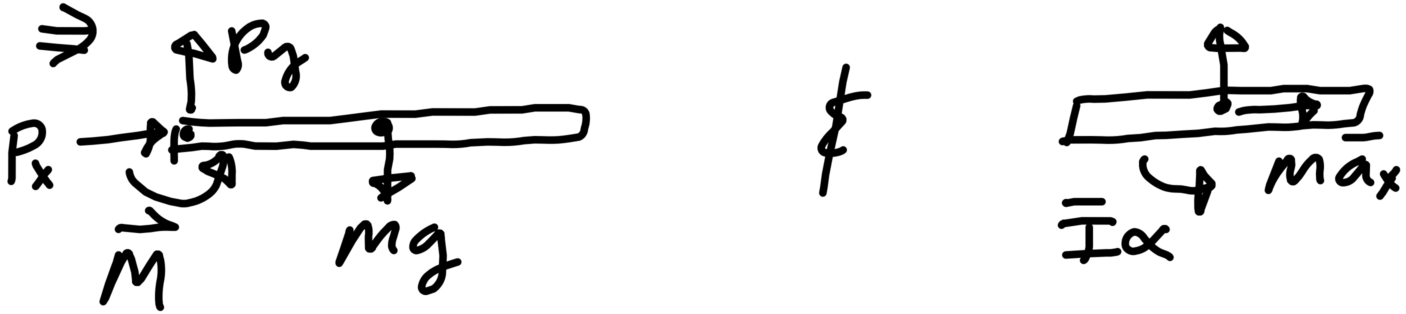


About G: $\circlearrowleft \Sigma M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha$ (1)

About P: $\circlearrowleft \Sigma M_P = \bar{I}\alpha + \vec{r}_{G/P} \times m\bar{\vec{a}} \Rightarrow M - \frac{L}{2}mg = \bar{I}\alpha + \frac{mL\bar{a}_y}{2}$

But $\Sigma F_y = m\bar{a}_y$

Example

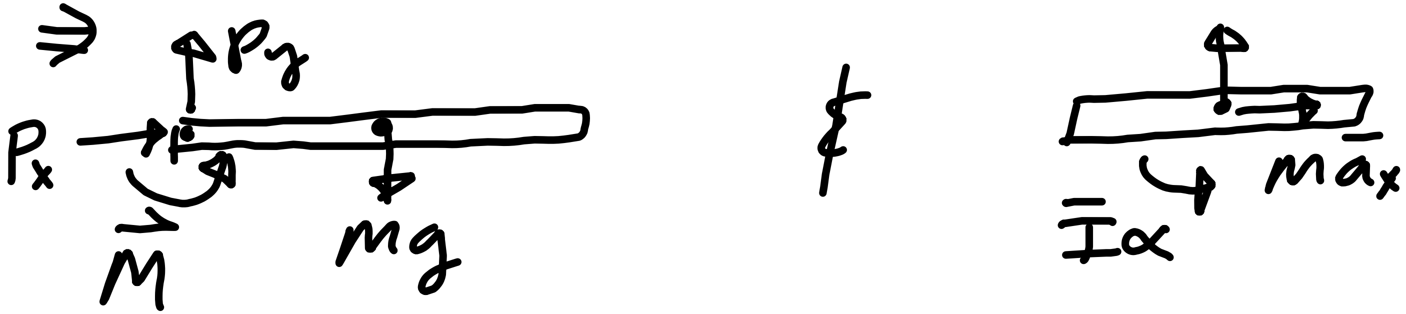


About G: $\circlearrowleft \sum M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha$ (1)

About P: $\circlearrowleft \sum M_P = \bar{I}\alpha + \vec{r}_{G/P} \times m\bar{\vec{a}} \Rightarrow M - \frac{L}{2}mg = \bar{I}\alpha + \frac{mL\bar{a}_y}{2}$

But $\sum F_y = m\bar{a}_y \Rightarrow P_y - mg = m\bar{a}_y$

Example



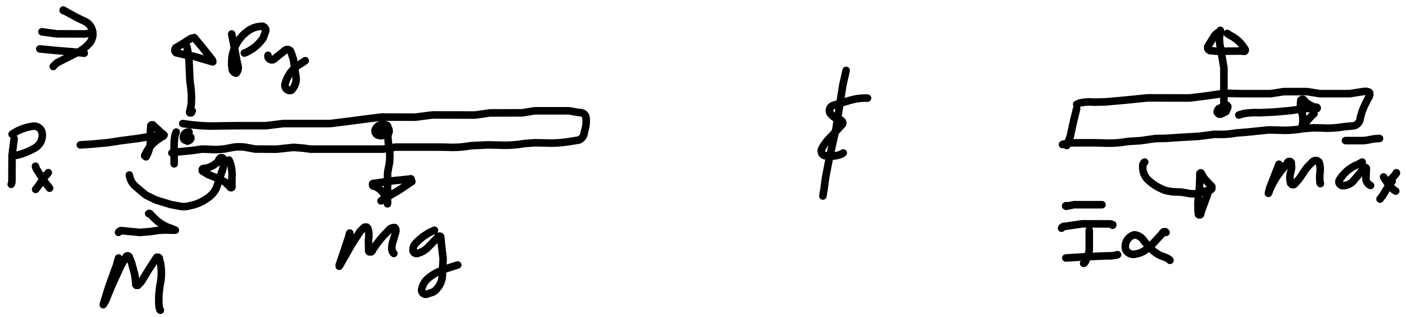
About G: $\circlearrowleft \Sigma M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2} P_y = \bar{I}\alpha \quad (1)$

About P: $\circlearrowleft \Sigma M_P = \bar{I}\alpha + \vec{r}_{G/P} \times m\bar{\vec{a}} \Rightarrow M - \frac{L}{2} mg = \bar{I}\alpha + \frac{mL\bar{a}_y}{2}$

But $\Sigma F_y = m\bar{a}_y \Rightarrow P_y - mg = m\bar{a}_y$

$\Rightarrow M - \frac{L}{2} mg = \bar{I}\alpha + \frac{L}{2}(P_y - mg)$

Example

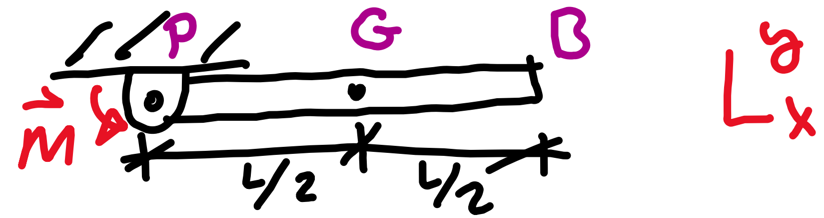


About G: $\Sigma M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha$ (1)

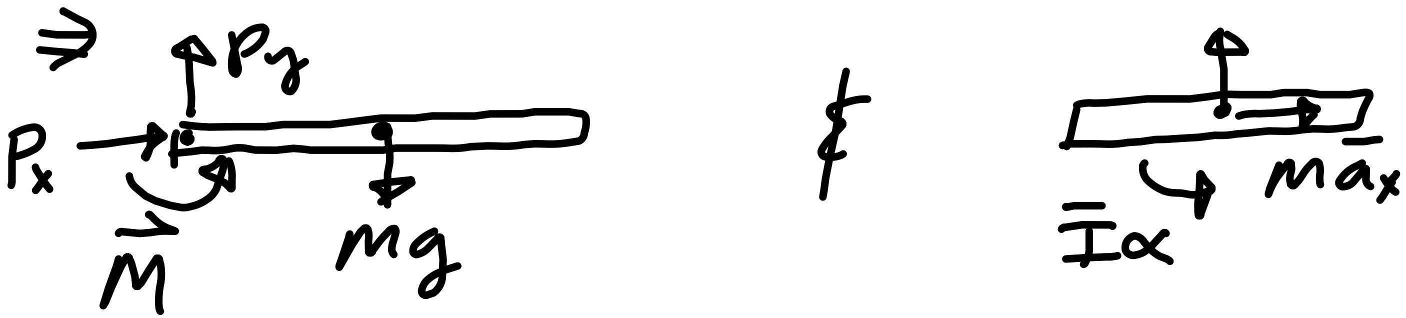
About P: $\Sigma M_P = \bar{I}\alpha + \vec{r}_{G/P} \times m\bar{a} \Rightarrow M - \frac{L}{2}mg = \bar{I}\alpha + \frac{mL\bar{a}_y}{2}$

But $\Sigma F_y = m\bar{a}_y \Rightarrow P_y - mg = m\bar{a}_y$

$\Rightarrow M - \frac{L}{2}mg = \bar{I}\alpha + \frac{L}{2}(P_y - mg) \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha$



Example



About G: $\Sigma M_G = \bar{I}\alpha \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha$ (1)

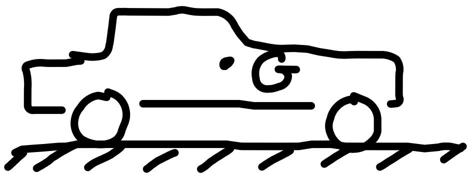
About P: $\Sigma M_P = \bar{I}\alpha + \vec{r}_{G/P} \times m\bar{\vec{a}} \Rightarrow M - \frac{L}{2}mg = \bar{I}\alpha + \frac{mL\bar{a}_y}{2}$

But $\Sigma F_y = m\bar{a}_y \Rightarrow P_y - mg = m\bar{a}_y$

$$\Rightarrow M - \frac{L}{2}mg = \bar{I}\alpha + \frac{L}{2}(P_y - mg) \Rightarrow \boxed{M - \frac{L}{2}P_y = \bar{I}\alpha}$$

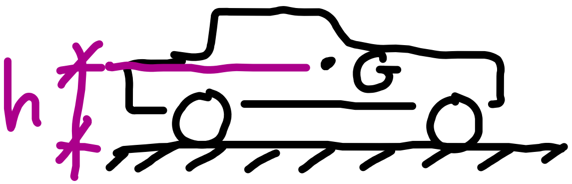
Same as equation 1

Notes on 16.3



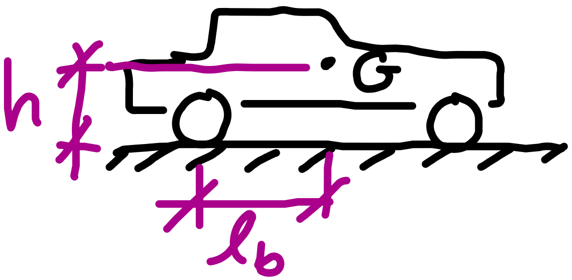
Notes on 16.3

$h = 20\,in$



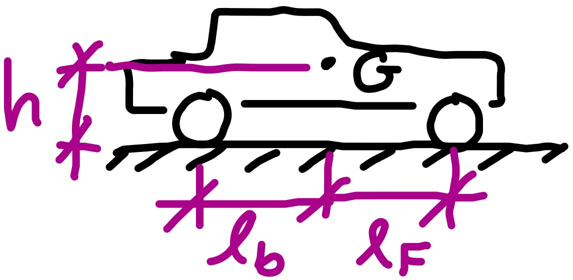
Notes on 16.3

$h = 20\text{in}, \ell_b = 60\text{in}$

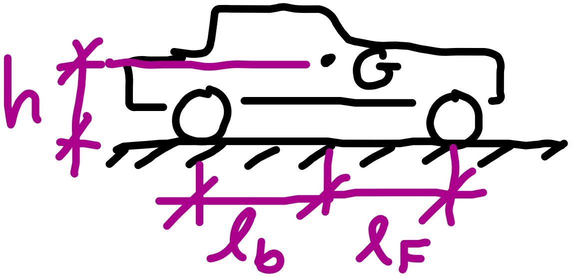


Notes on 16.3

$h = 20in, l_b = 60in, l_F = 40in$



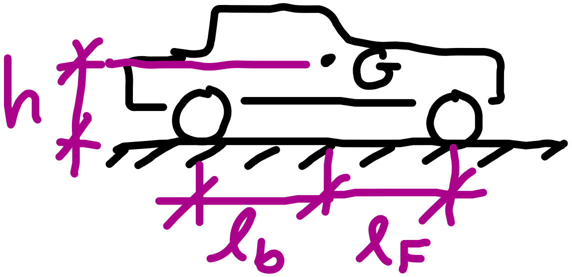
Notes on 16.3



$h = 20\,in$, $\ell_b = 60\,in$, $\ell_F = 40\,in$

$\mu_K = 0.80$

Notes on 16.3



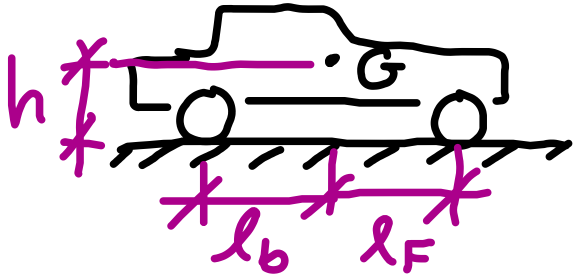
$h = 20\text{in}, \; l_b = 60\text{in}, \; l_F = 40\text{in}$

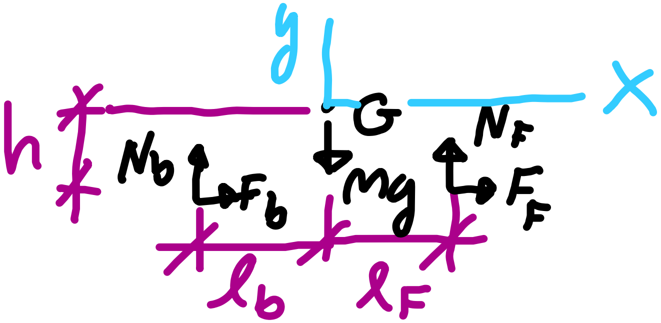
$\mu_K = 0.80$ Find a_{MAX}:

Notes on 16.3

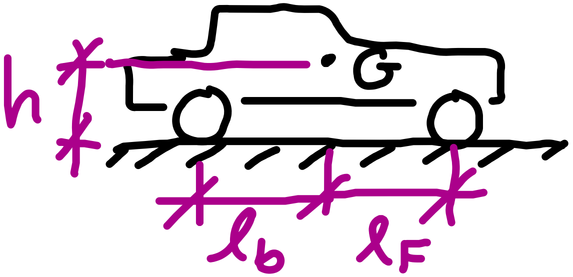
$h = 20in, \ell_b = 60in, \ell_F = 40in$

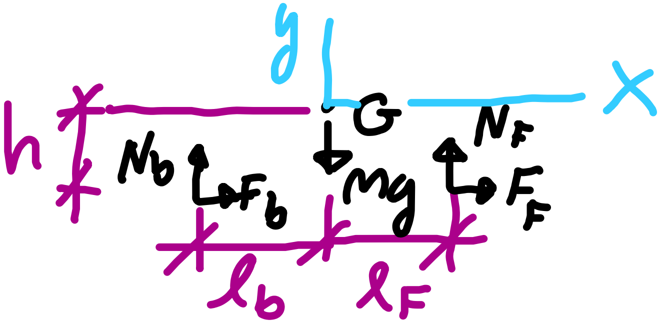
$\mu_K = 0.80$ Find a_{MAX}:





Notes on 16.3



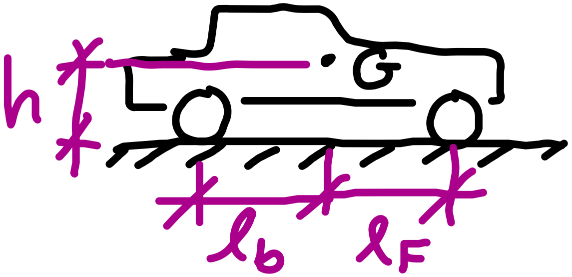


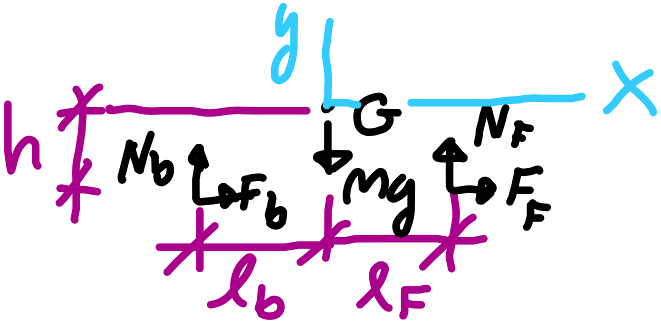
$h = 20in$, $\ell_b = 60in$, $\ell_F = 40in$

$\mu_K = 0.80$ **Find a_{MAX}:**

4-wheel drive:

Notes on 16.3





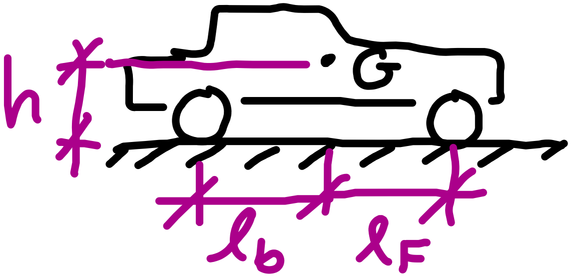
$h = 20 in, l_b = 60 in, l_F = 40 in$

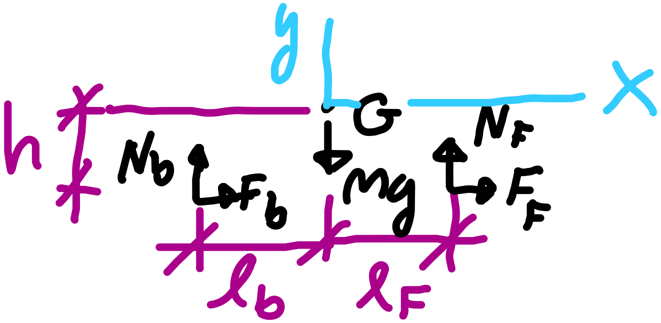
$\mu_K = 0.80$ Find a_{MAX}:

4-wheel drive:

$$\Sigma F_y = 0 \Rightarrow N_B + N_F = mg$$

Notes on 16.3





$h = 20in$, $l_b = 60in$, $l_F = 40in$

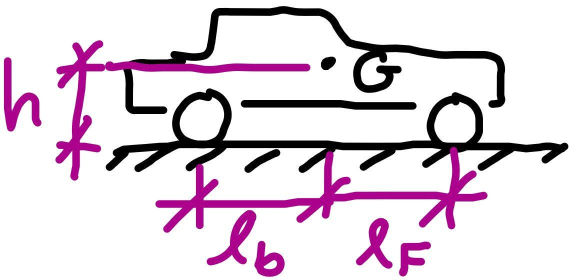
$\mu_K = 0.80$ Find a_{MAX}:

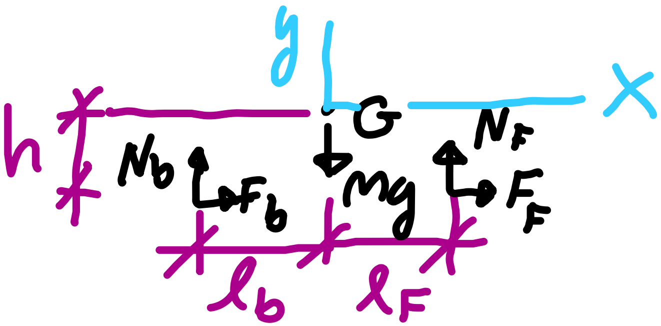
4-wheel drive:

$$\Sigma F_y = 0 \Rightarrow N_B + N_F = mg$$

$$\overset{+}{\rightarrow} \Sigma F_x = N_B \mu_s + N_F \mu_s$$

Notes on 16.3





$h = 20in$, $\ell_b = 60in$, $\ell_F = 40in$

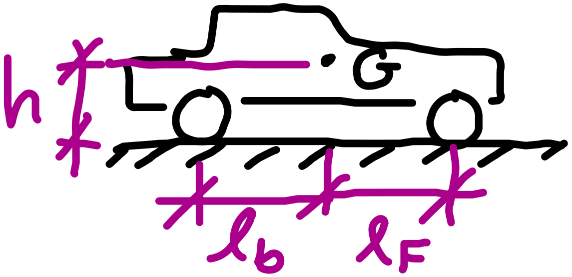
$\mu_K = 0.80$ Find a_{max}:

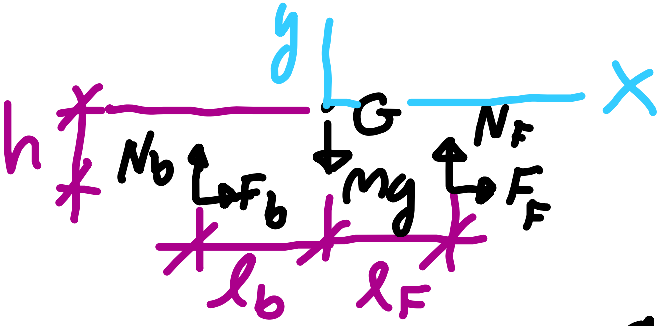
4-wheel drive:

$$\sum F_y = 0 \Rightarrow N_B + N_F = mg$$

$$\overset{+}{\leftarrow} \sum F_x = N_B \mu_s + N_F \mu_s = m a_x$$

Notes on 16.3





$h = 20\,in,\ l_b = 60\,in,\ l_F = 40\,in$

$\mu_K = 0.80$ Find a_{max} :

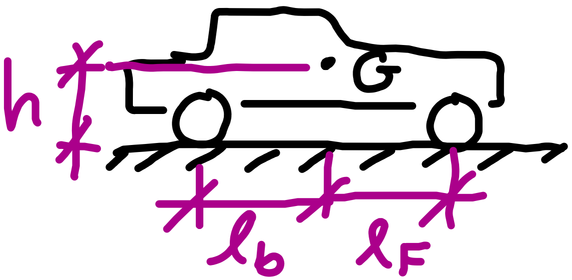
4-wheel drive:

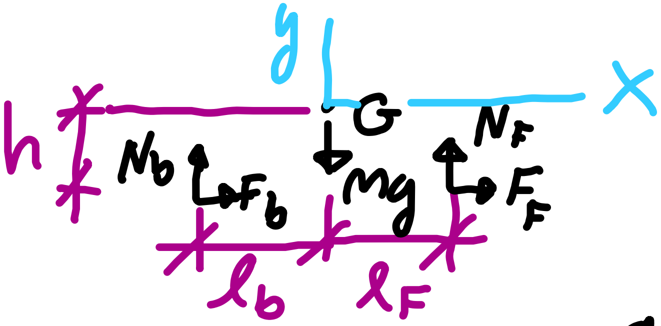
$$\Sigma F_y = 0 \Rightarrow N_B + N_F = mg$$

$$\& \ \Sigma F_x = N_B \mu_s + N_F \mu_s = m a_x$$

so just solve for a_x & no need for moment analysis

Notes on 16.3





$h = 20\text{in}$, $l_b = 60\text{in}$, $l_F = 40\text{in}$

$\mu_K = 0.80$ Find a_{max}:

4-wheel drive:

$$\sum F_y = 0 \Rightarrow N_B + N_F = mg$$

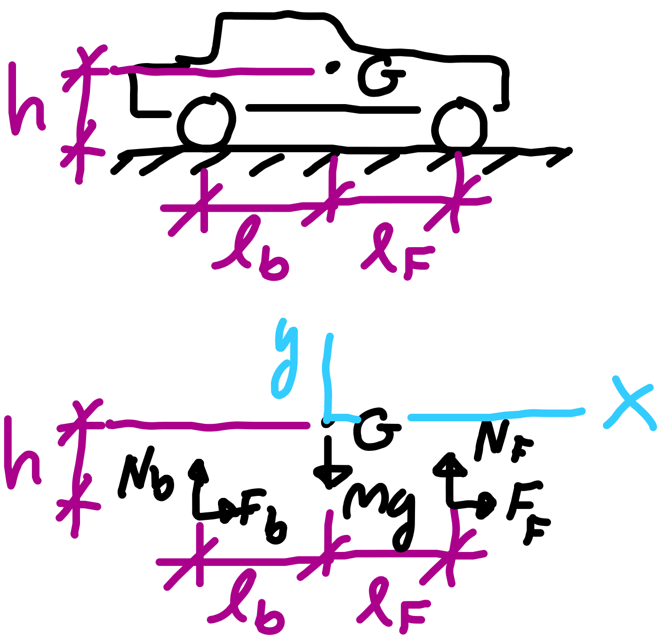
& $\sum F_x = N_B \mu_s + N_F \mu_s = m a_x$

so just solve for a_x & no need for moment analysis ☺

Notes on 16.3

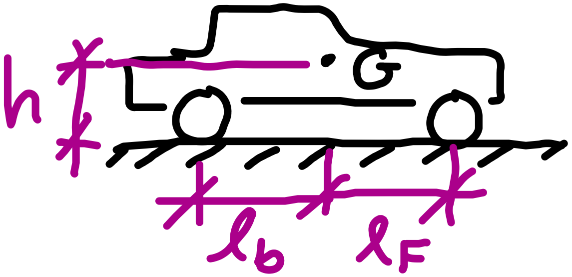
$h = 20in$, $l_b = 60in$, $l_F = 40in$

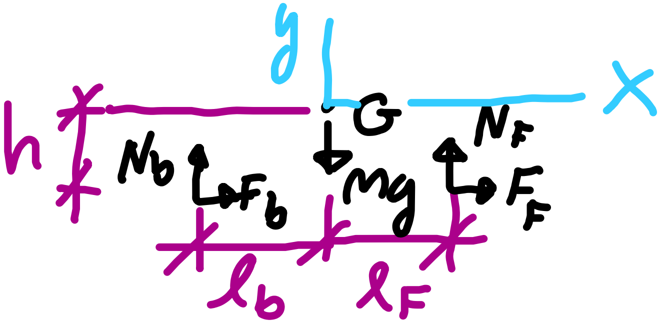
$\mu_K = 0.80$ <u>Find a_{MAX}</u> :



Front wheel drive :

Notes on 16.3





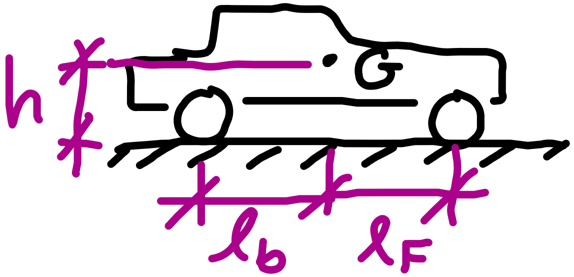
$h = 20in$, $\ell_b = 60in$, $\ell_F = 40in$

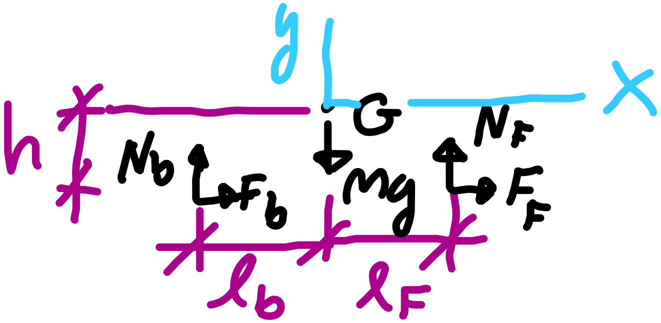
$\mu_K = 0.80$ Find a_{MAX}:

Front wheel drive:

NOTE: Rear wheel drive analysis is similar

Notes on 16.3





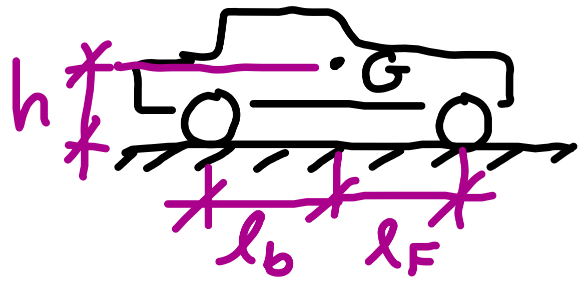
$h = 20\text{in}, \ l_b = 60\text{in}, \ l_F = 40\text{in}$

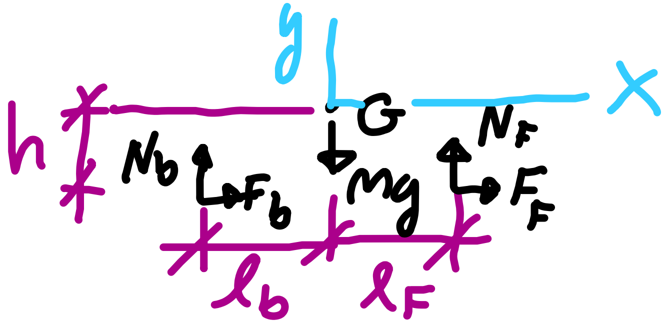
$\mu_K = 0.80$ Find a_{MAX} :

Front wheel drive :

$F_b = 0$ & $F_F \neq 0$

Notes on 16.3





$h = 20in$, $l_b = 60in$, $l_F = 40in$

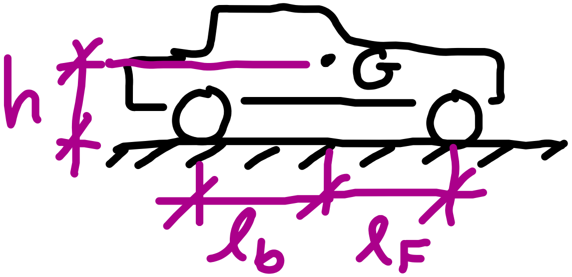
$\mu_K = 0.80$ Find a_{MAX}:

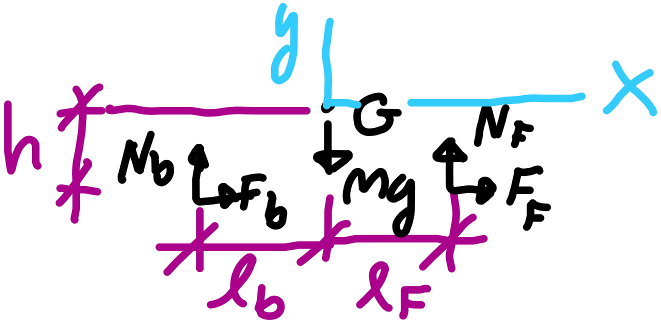
Front wheel drive:

$F_b = 0$ & $F_F \neq 0$

Here $\sum F_x = ma_x$

Notes on 16.3





$h = 20in$, $l_b = 60in$, $l_F = 40in$

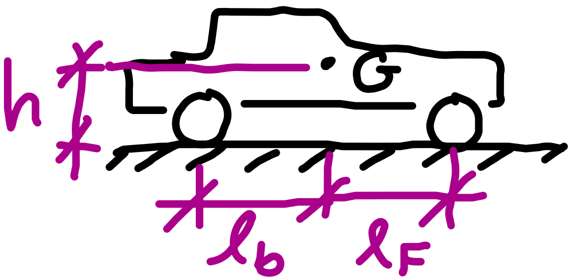
$\mu_K = 0.80$ <u>Find a_{MAX}</u>:

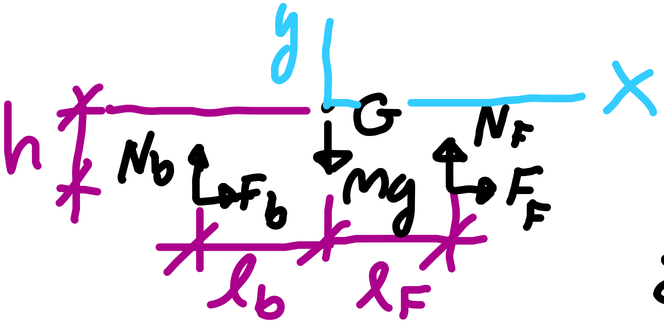
<u>Front wheel drive:</u>

$F_b = 0$ & $F_F \neq 0$

Here $\Sigma F_x = ma_x \Rightarrow N_F \mu_s = ma_x$ (1)

Notes on 16.3





$h = 20\text{in}$, $l_b = 60\text{in}$, $l_F = 40\text{in}$

$\mu_K = 0.80$ Find a_{MAX}:

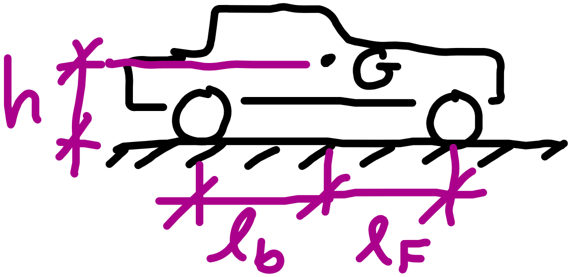
Front wheel drive:

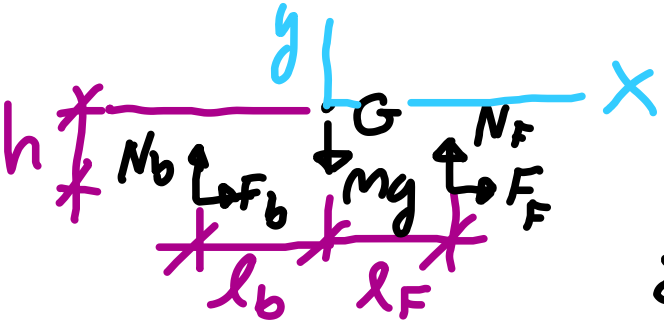
$F_b = 0$ & $F_F \neq 0$

Here $\sum F_x = ma_x \Rightarrow N_F \mu_s = ma_x$ (1)

& $\sum F_y = 0$

Notes on 16.3





$h = 20in$, $l_b = 60in$, $l_F = 40in$

$\mu_K = 0.80$ Find a_{MAX}:

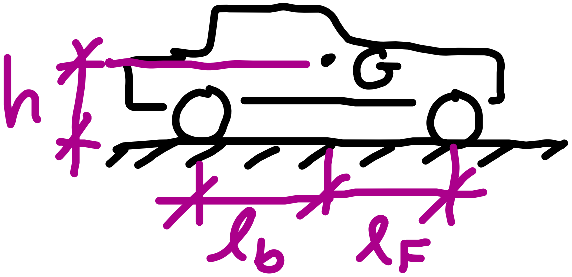
Front wheel drive:

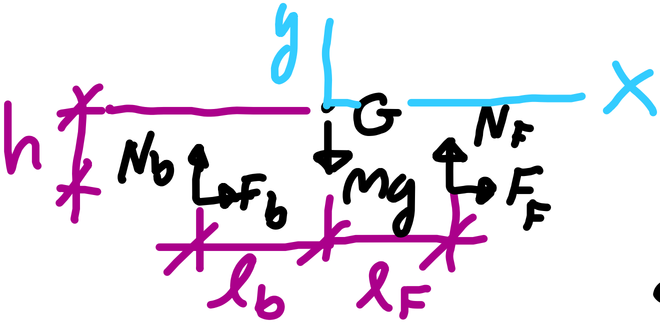
$F_b = 0$ & $F_F \neq 0$

Here $\Sigma F_x = m a_x \Rightarrow N_F \mu_s = m a_x^{(1)}$

& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg^{(2)}$

Notes on 16.3





$h = 20in$, $l_b = 60in$, $l_F = 40in$

$\mu_K = 0.80$ **Find a_{max}:**

Front wheel drive:

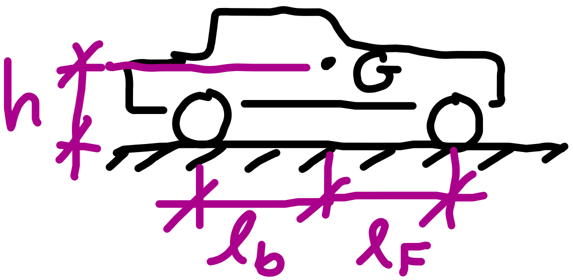
$F_b = 0$ & $F_F \neq 0$

Here $\Sigma F_x = ma_x \Rightarrow N_F \mu_s = ma_x$ $^{(1)}$

& $\Sigma F_y = 0 \Rightarrow \boxed{N_B + N_F = mg}$ $^{(2)}$

As before

Notes on 16.3

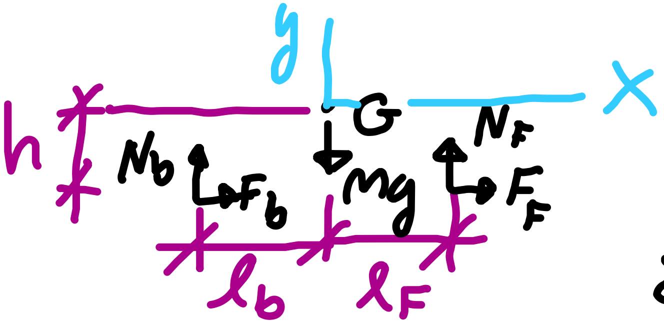


$h = 20in,\ l_b = 60in,\ l_F = 40in$

$\mu_K = 0.80$ Find a_{MAX}:

Front wheel drive:

$F_b = 0$ & $F_F \neq 0$

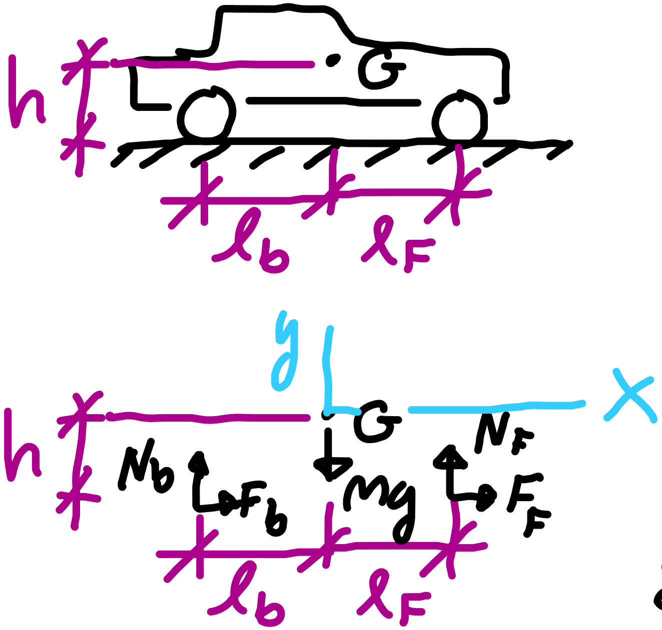


Here $\Sigma F_x = ma_x \Rightarrow N_F \mu_s = ma_x^{(1)}$

& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg^{(2)}$

Now need moment analysis.

Notes on 16.3



$h = 20 in, l_b = 60 in, l_F = 40 in$

$\mu_K = 0.80$ Find a_{MAX}:

Front wheel drive:

$F_b = 0$ & $F_F \neq 0$

Here $\Sigma F_x = ma_x \Rightarrow N_F \mu_s = ma_x$ (1)

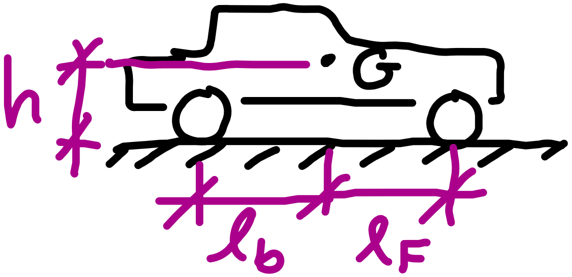
& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg$ (2)

Now need moment analysis. I am choosing point B (where rear wheels touch ground) as a reference point for the moment analysis

Notes on 16.3

$h = 20\,in,\ l_b = 60\,in,\ l_F = 40\,in$

$\mu_K = 0.80$ Find a_{MAX}:



Front wheel drive:

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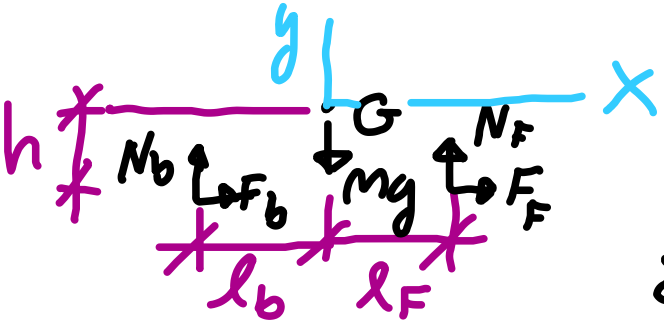


Here $\Sigma F_x = m a_x \Rightarrow N_F \mu_s = m a_x^{(1)}$

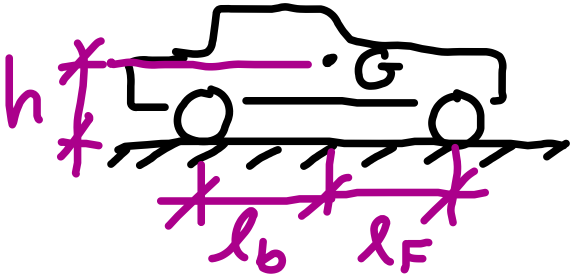
& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg^{(2)}$

$\circlearrowleft \Sigma M_B = I\vec{\alpha} + \vec{r}_{G/B} \times \vec{a}$

Notes on 16.3

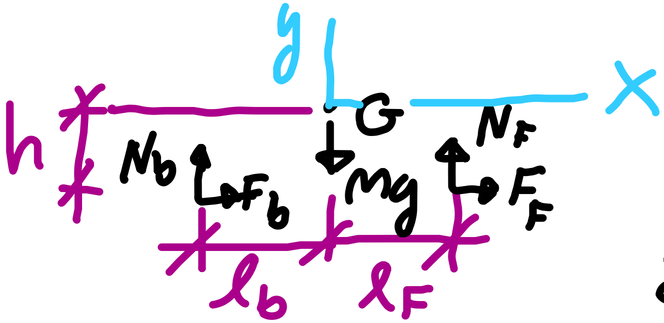
$h=20in, \; l_b=60in, \; l_F=40in$

$\mu_K=0.80$ Find a_{MAX}:



Front wheel drive:

$F_b=0$ & $F_F \neq 0$

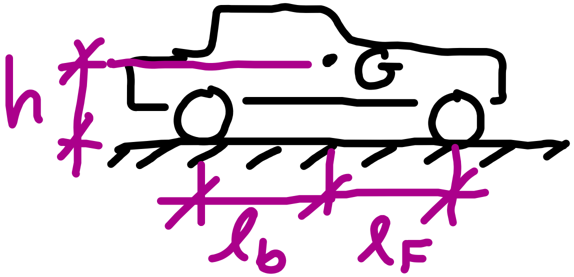


Here $\sum F_x = m a_x \Rightarrow N_F \mu_s = m a_x^{(1)}$

& $\sum F_y = 0 \Rightarrow N_B + N_F = mg^{(2)}$

$\circlearrowleft \sum M_B = I\vec{\alpha} + \vec{r}_{G/B} \times \vec{a}$ Here $\alpha = 0$

Notes on 16.3



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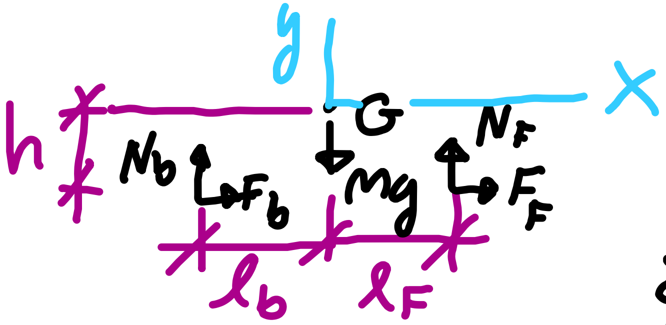

Here $\Sigma F_x = ma_x \Rightarrow N_F\mu_s = ma_x$ (1)

& $\Sigma F_y = 0 \Rightarrow N_B+N_F = mg$ (2)

$\circlearrowleft \Sigma M_B = I\vec{\alpha} + \vec{r}_{G/B}\times\vec{\bar{a}}$ Here $\alpha=0$ &

$\vec{r}_{G/B}\times\vec{\bar{a}} = h\bar{a}_x$

Notes on 16.3

$h = 20\,in,\ \ell_b = 60\,in,\ \ell_F = 40\,in$

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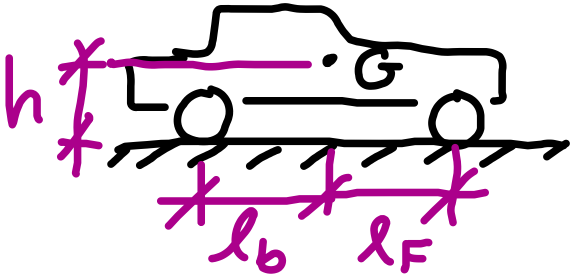

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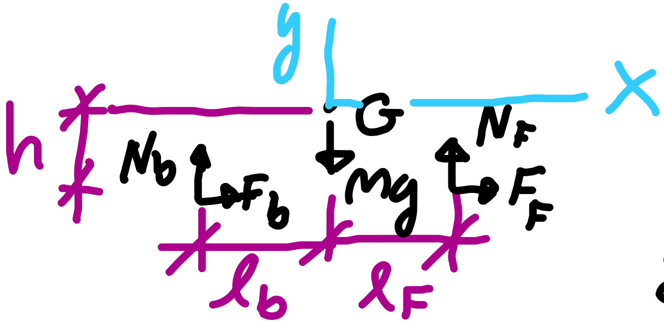

Here $\Sigma F_x = m a_x \Rightarrow N_F \mu_s = m a_x$ (1)

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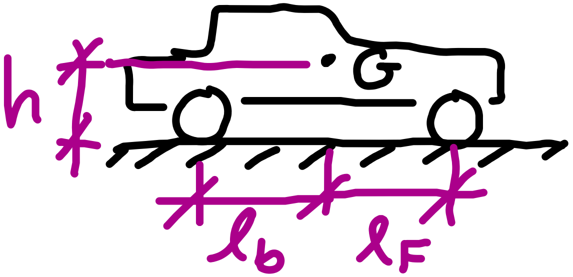
Notes on 16.3

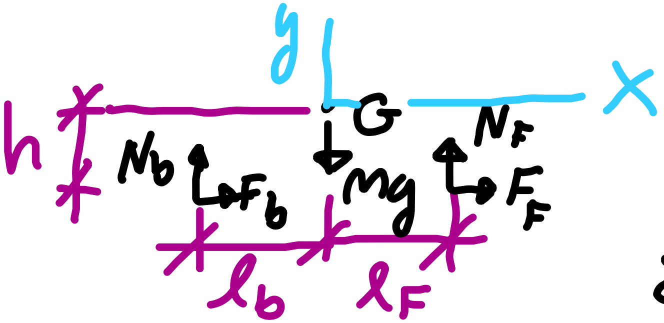


$h = 20in, \, l_b = 60in, \, l_F = 40in$

$\mu_K = 0.80$ Find a_{MAX}:

Front wheel drive:

$F_b = 0$ & $F_F \neq 0$



Here $\Sigma F_x = ma_x \Rightarrow N_F \mu_s = ma_x$ ⁽¹⁾

& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg$ ⁽²⁾

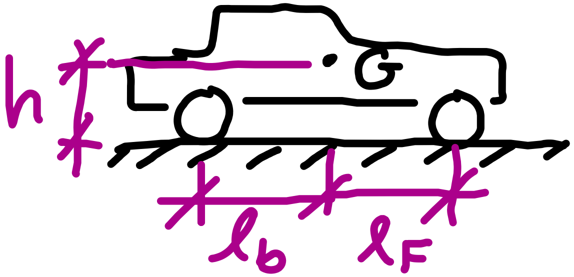
$\circlearrowleft \Sigma M_B = I\vec{\alpha} + \vec{r}_{G/B} \times \vec{a}$ Here $\alpha = 0$ &

$\vec{r}_{G/B} \times \vec{a} = h\,\bar{a}_x$ & $\vec{M}_B = [l_b mg - (l_b + l_F)N_F] \circlearrowleft$

Notes on 16.3

$h = 20\,in,\ \ell_b = 60\,in,\ \ell_F = 40\,in$

$\mu_K = 0.80$ **Find a_{MAX}:**



Front wheel drive:

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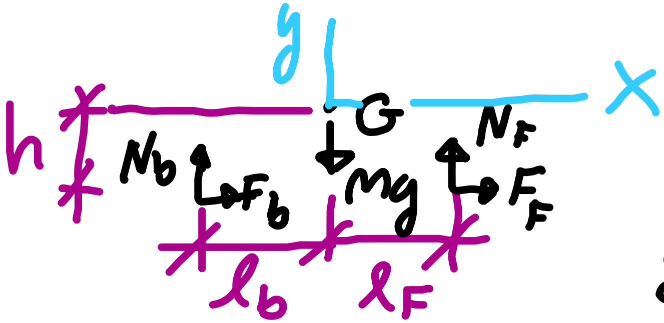

Here $\Sigma F_x = m a_x \Rightarrow N_F \mu_s = m a_x^{(1)}$

& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg^{(2)}$

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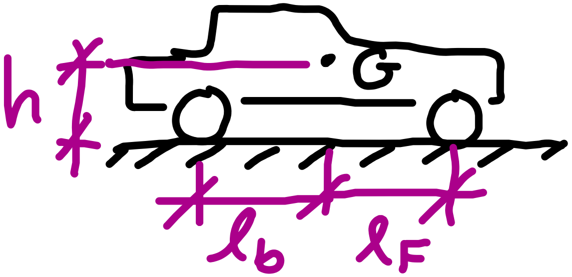
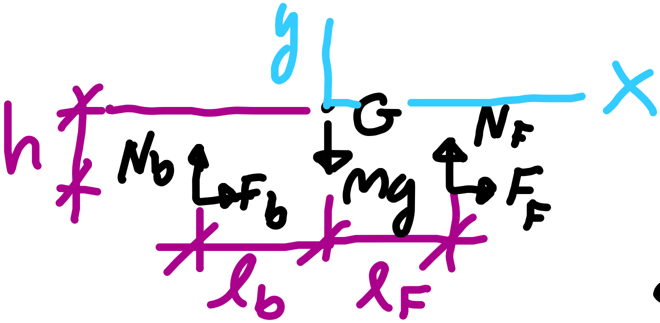
$\vec{r}_{G/B} \times \vec{\bar{a}} = h\,\bar{a}_x$ & $\vec{M}_B = [\ell_b mg - (\ell_b + \ell_F)N_F] \circlearrowright$

Notes on 16.3

$h=20\text{in}, \ l_b=60\text{in}, \ l_F=40\text{in}$

$\mu_K=0.80$ Find a_{max} :





Frontwheel drive:

$F_b = 0$ & $F_F \neq 0$

Here $\Sigma F_x = ma_x \Rightarrow N_F \mu_s = ma_x$ $^{(1)}$

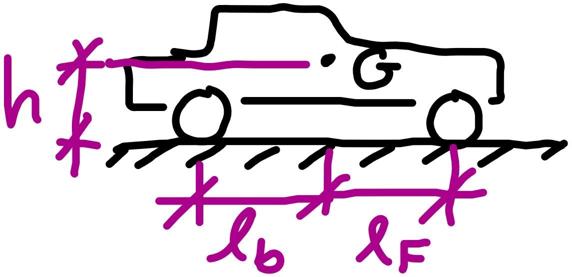
& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg$ $^{(2)}$

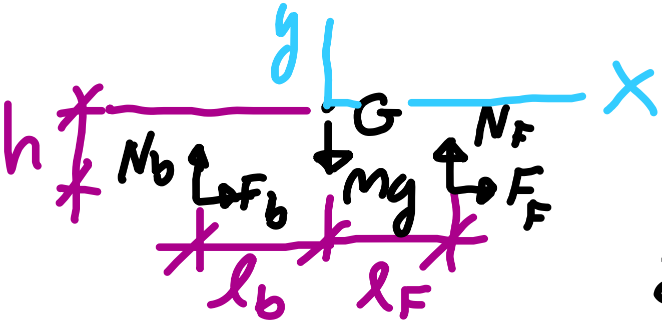
$\circlearrowleft \Sigma M_B = I\vec{\alpha} + \vec{r}_{G/B} \times \vec{\bar{a}}$ Here $\alpha = 0$ &

$\vec{r}_{G/B} \times \vec{\bar{a}} = h\bar{a}_x$ & $\vec{M}_B = [l_b mg - (l_b + l_F)N_F] \circlearrowleft$

But eqn 1 says $N_F = \frac{m}{\mu_s} a_x$

Notes on 16.3





$h = 20in,\ l_b = 60in,\ l_F = 40in$

$\mu_K = 0.80$ Find a_{MAX}:

Front wheel drive:

$F_b = 0$ & $F_F \neq 0$

Here $\Sigma F_x = m a_x \Rightarrow N_F \mu_s = m a_x^{(1)}$

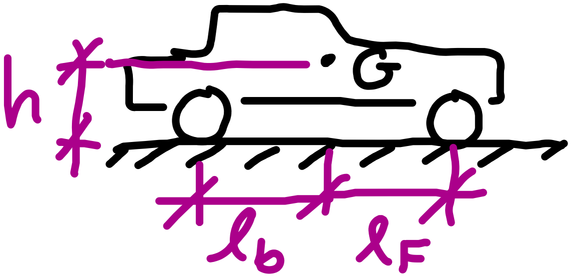
& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg^{(2)}$

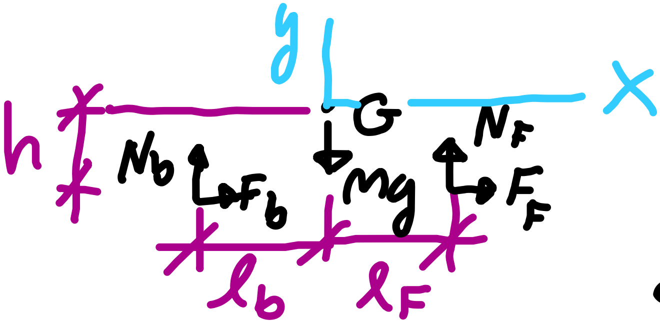
$\circlearrowleft \Sigma M_B = I\vec{\alpha} + \vec{r}_{G/B} \times \vec{\bar{a}}$ Here $\alpha = 0$ &

$\vec{r}_{G/B} \times \vec{\bar{a}} = h\,\bar{a}_x$ & $\vec{M}_B = [l_b mg - (l_b + l_F) N_F] \circlearrowleft$

So $l_b mg - (l_b + l_F)\left(\frac{m a_x}{\mu_s}\right) = m h \bar{a}_x$

Notes on 16.3





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So $l_b mg - (l_b + l_F)\left(\frac{m a_x}{\mu_s}\right) = m h \bar{a}_x \Rightarrow a_x\left(h + \frac{l_b + l_F}{\mu_s}\right) = l_b g$

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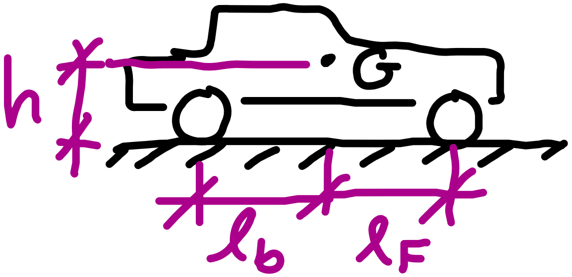

Front wheel drive:

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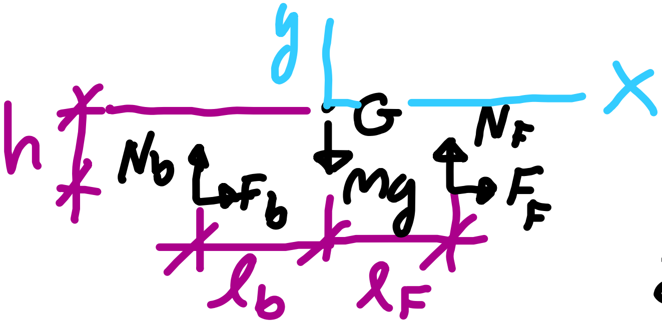

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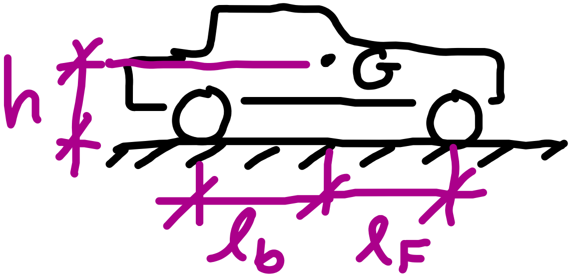
$\curvearrowleft \sum M_B = I\vec{\alpha} + \vec{r}_{G/B} \times \vec{a}$ Here $\alpha = 0$ &

$\vec{r}_{G/B} \times \vec{a} = h\bar{a}_x$ & $\vec{M}_B = [\ell_b mg - (\ell_b + \ell_F) N_F] \circlearrowright$

So $\ell_b mg - (\ell_b + \ell_F)\left(\frac{m a_x}{\mu_s}\right) = m h \bar{a}_x \Rightarrow a_x\left(h + \frac{\ell_b + \ell_F}{\mu_s}\right) = \ell_b g$

$$\Rightarrow a_x = \frac{\ell_b g}{h + (\ell_b + \ell_F)/\mu_s}$$

Notes on 16.3



$h = 20 \text{in}, \ell_b = 60 \text{in}, \ell_F = 40 \text{in}$

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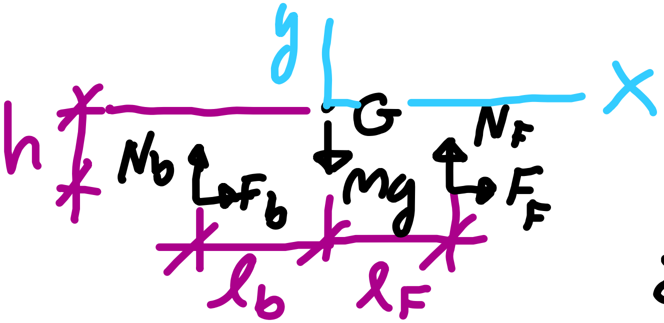

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So $\ell_b mg - (\ell_b + \ell_F)\left(\frac{m a_x}{\mu_s}\right) = m h \bar{a}_x \Rightarrow a_x\left(h + \frac{\ell_b + \ell_F}{\mu_s}\right) = \ell_b g$

$$\Rightarrow a_x = \frac{\ell_b g}{h + (\ell_b + \ell_F)/\mu_s} = \left[\frac{60 * 32.2 \text{ ft/s}^2}{20 + 100/0.8}\right]$$

Notes on 16.3

$h = 20\,in,\ l_b = 60\,in,\ l_F = 40\,in$

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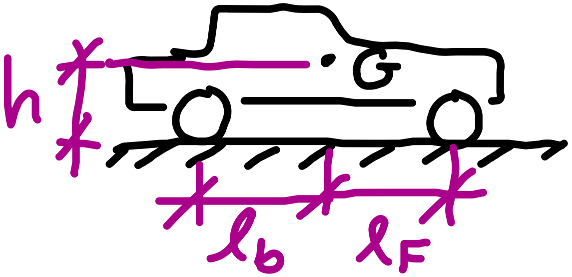

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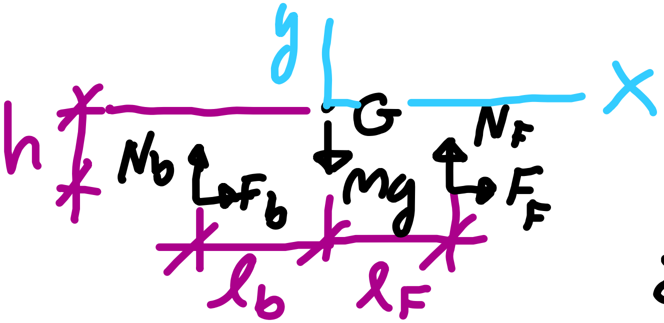

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So $l_b mg - (l_b + l_F)\left(\frac{m a_x}{\mu_s}\right) = m h \bar{a}_x \Rightarrow a_x\left(h + \frac{l_b + l_F}{\mu_s}\right) = l_b g$

$$\Rightarrow \quad a_x = \frac{l_b g}{h + (l_b + l_F)/\mu_s} = \left[\frac{60 * 32.2\ ft/s^2}{20 + 100/0.8}\right] = 13.32\ \frac{ft}{s^2}$$